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The long time behavior of Brownian motion in tilted periodic potentials

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HIGHLIGHTS

- We study the Langevin equation describing diffusion in tilted periodic potentials.
- In the over-damped limit the long time average velocity converges in probability.
- In the over-damped and vanishing noise limit the convergence rate varies as the tilt crosses threshold.
- In the under-damped limit we recover Risken's results about the bi-stability.
- In the under-damped limit we derive asymptotics of the transition times.

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ABSTRACT

A variety of phenomena in physics and other fields can be modeled as Brownian motion in a heat bath under tilted periodic potentials. We are interested in the long time average velocity considered as a function of the external force, that is, the tilt of the potential. In many cases, the long time behavior – pinning and de-pinning phenomenon – has been observed. We use the method of stochastic differential equation to study the Langevin equation describing such diffusion. In the *over-damped limit*, we show the convergence of the long time average velocity to that of the Smoluchowski–Kramers approximation, and carry out asymptotic analysis based on Risken's and Reimann et al.'s formula. In the *under-damped limit*, applying Freidlin et al.'s theory, we first show the existence of three pinning and de-pinning thresholds of the normalized tilt, corresponding to the bi-stability phenomenon; and second, as noise approaches zero, derive formulas of the mean transition times between the pinning and running states.

1. Introduction

We are concerned with the *pinning* and *de-pinning* behavior of particles moving in inhomogeneous materials when a *driving force* F crosses over some threshold value F^*. That is, as shown in Fig. 1, there exists a threshold F^* of the driving force F such that if F is less than F^*, the *long time average velocity* of the particle V_F (considered as a function of F) is *zero*; whilst if F is greater than F^*, V_F is *positive*.

The study of pinning and de-pinning phenomenon is motivated by many applications, for example, the observations regarding charge-density waves at low temperatures and the concomitant nonlinear conductivity characterized by non-Ohmic behavior above a small threshold electric field. It is believed that the non-Ohmic conduction is caused by the sliding of charge-density wave which is prevented from moving below the threshold field by pinning to impurities and other lattice defects [1]. The pinning effect due to the crystal defects or impurities can also be observed frequently in a type II superconductor with impurities. Analogous pinning phenomena in a Josephson junction – an array of superconductors separated by a thin insulating barrier – in the presence of different types of structural disorder is investigated both analytically and numerically [2–5]. In addition, the pinning and de-pinning (stick–slip) character in the motion of a phase boundary leads to the widely observed rate-independent hysteresis feature in shape-memory alloys [6]. This phenomenon is also related to systems such as dynamics of cracks and geological faults, which can be modeled as front propagation describing the evolution of an interface driven by an external force through an inhomogeneous medium [7,8].

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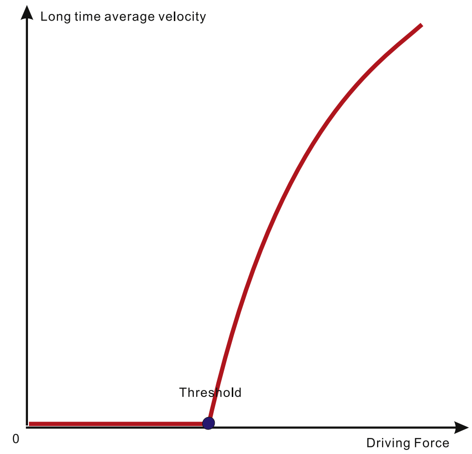

Fig. 1. Pinning and de-pinning phenomena.

In order to obtain more quantitative information about dynamical behaviors, we consider a one dimensional model of particle motion in a *periodic* medium. This model has been applied to study the dynamics of a de-pinned, uniform charge-density wave [9]. We analyze the dynamics in both the *over-damped (dissipation driven)* and *under-damped (inertia driven)* limits. A specific example, in the one-dimensional deterministic case, is given by the following equation:

$$\dot{x} = -\Psi'(x) + F, \qquad x(0) = x_0, \tag{1.1}$$

where $x(\cdot)$ is the position of a particle which evolves on a periodic potential function Ψ. The constant F signifies an external forcing. The *long time average velocity* V_F is defined as

$$V_F := \lim_{t\to\infty} \frac{x(t)}{t}. \tag{1.2}$$

For the above example, if Ψ satisfies some *non-degenerate* condition, then we have [1,9,6,10]:

$$V_F \sim C(F - F^*)^{\frac{1}{2}} \quad \text{for } 0 < F - F^* \ll 1. \tag{1.3}$$

(The above scaling will be explained in page section.)

The goal of this paper is to extend the understanding of (1.1) and (1.3) to incorporate stochastic noise and inertial effects. For this purpose, the following second order *Langevin equation* [11,12] which is an analogue of Newton's Second Law, is often considered:

$$m\ddot{q} = F - \nabla\Psi(q) - \gamma\dot{q} + \sqrt{2\gamma\beta^{-1}}\dot{W}, \qquad q(0) = q_0, \qquad \dot{q}(0) = p_0, \tag{1.4}$$

where the position variable q is a d-dimensional vector in $\mathbb{R}^d$, m is the mass of the particle, F is the driving force, Ψ denotes a smooth periodic potential function depending on the position variable q, γ is the damping coefficient, $\beta = (kT)^{-1}$ is the inverse temperature (k is the Boltzmann constant and T is the absolute temperature), and $\dot{W}$ is a standard d-dimensional white noise. Note that the *fluctuation–dissipation* criterion is imposed in the above form of the equation. It is often useful to consider the above equation in the form of a first order system:

$$\dot{q} = p, \qquad \dot{p} = \frac{F - \nabla\Psi(q)}{m} - \frac{\gamma}{m}p + \frac{\sqrt{2\gamma\beta^{-1}}}{m}\dot{W}, \qquad q(0) = q_0, \qquad p(0) = p_0, \tag{1.5}$$

where q and p are the position and velocity variables. We want to study the *long time average velocity* of the particle diffusion described by (1.4) or (1.5) as a *function of the driving force F* in various regimes. In order to obtain more quantitative results we restrict ourselves to one dimensional case, i.e., $d = 1$. We first give an outline of our results.

For the **over-damped limit**, we will state the convergence of the long time average velocity in the limits of *vanishing noise* and *vanishing mass* in Sections 3 and 4.

- In Section 3, we concentrate on the following *first order* equation

$$\gamma\dot{q} = F - \Psi'(q) + \sqrt{2\gamma\beta^{-1}}\dot{W}, \qquad q(0) = q_0, \tag{1.6}$$

  which is often called the *Smoluchowski* equation. We show in Theorem 3.1 that V_F converges to its deterministic version as the noise goes to zero ($\beta \to \infty$). The convergence rate is exponential (in β) when the driving force is below the pinning and de-pinning threshold (Region I), but only algebraic when the driving force is at or above the threshold (Regions II and III). This is illustrated in Fig. 2. The main technique used is Laplace's method applied to an explicit formula for V_F.
- In Section 4, we will consider the second order Langevin equation (1.5) which incorporates inertia effects. We show the convergence of V_F in the vanishing mass limit ($m \to 0$) in both the deterministic and stochastic versions. For the deterministic case (Theorem 4.1), we make use of the classifications of the ω-limit sets of the dynamics (see for example [13]). For the stochastic case (Theorem 4.2) we make use of the ergodicity of systems (1.5) and (1.6). The ergodicity of the former is nontrivial, since the system is driven by a *degenerate noise* — the noise is directly applied only to the velocity variable $\dot{q}$ (but not the spatial variable q). This is proved by M. Hairer and G.A. Pavliotis in [14] which makes use of the *hypoellipticity* of the Fokker–Planck operator for system (1.5).