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Complex short pulse and coupled complex short pulse equations

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HIGHLIGHTS

- We propose two model equations to describe ultra-short pulses in nonlinear optics.
- We show the integrability by providing the Lax pairs and constructing local and nonlocal conservation laws.
- We construct multi-soliton solutions in pfaffians by Hirota's bilinear method.
- One- and two-soliton solutions are investigated in details, which show many interesting properties.

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ABSTRACT

In the present paper, we propose a complex short pulse equation and a coupled complex short equation to describe ultra-short pulse propagation in optical fibers. They are integrable due to the existence of Lax pairs and infinite number of conservation laws. Furthermore, we find their multi-soliton solutions in terms of pfaffians by virtue of Hirota's bilinear method. One- and two-soliton solutions are investigated in details, showing favorable properties in modeling ultra-short pulses with a few optical cycles. Especially, same as the coupled nonlinear Schrödinger equation, there is an interesting phenomenon of energy redistribution in soliton interactions. It is expected that, for the ultra-short pulses, the complex and coupled complex short pulses equation will play the same roles as the nonlinear Schrödinger equation and coupled nonlinear Schrödinger equation.

1. Introduction

The nonlinear Schrödinger (NLS) equation, as one of the universal equations that describe the evolution of slowly varying packets of quasi-monochromatic waves in weakly nonlinear dispersive media, has been very successful in many applications such as nonlinear optics and water waves [1–4]. The NLS equation is integrable, which can be solved by the inverse scattering transform [5].

However, in the regime of ultra-short pulses where the width of optical pulse is in the order of femtosecond (10^{-15} s), the NLS equation becomes less accurate [6]. Description of ultra-short processes requires a modification of standard slow varying envelope models based on the NLS equation. There are usually two approaches to meet this requirement in the literature. The first one is to add several higher-order dispersive terms to get higher-order NLS equation [2]. The second one is to construct a suitable fit to the frequency-dependent dielectric constant $\varepsilon(\omega)$ in the desired spectral range. Several models have been proposed by this approach including the short-pulse (SP) equation [7–10].

Recently, Schäfer and Wayne derived a so-called short pulse (SP) equation [7]

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx}, \tag{1}$$

to describe the propagation of ultra-short optical pulses in nonlinear media. Here, $u = u(x, t)$ is a real-valued function, representing the magnitude of the electric field, the subscripts t and x denote partial differentiation. Apart from the context of nonlinear optics, the SP equation has also been derived as an integrable differential equation associated with pseudospherical surfaces [11]. The SP equation has been shown to be completely integrable [11–15]. The periodic and soliton solutions of the SP equation were found in [16–18]. The connection between the SP equation and the sine–Gordon equation through the hodograph transformation was clarified, and then the N-soliton solutions including multi-loop and multi-breather ones were given in [19,20] by using the Hirota bilinear method [21]. The integrable discretization of the SP equation

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was studied in [22], the geometric interpretation of the SP equation, as well as its integrable discretization, was given in [23]. The higher-order corrections to the SP equation was studied in [24] most recently.

Similar to the case of the NLS equation [25], it is necessary to consider its two-component or multi-component generalizations of the SP equation for describing the effects of polarization or anisotropy. As a matter of fact, several integrable coupled short pulse have been proposed in the literature [26–31]. Most recently, the bi-Hamiltonian structures for the above two-component SP equations were obtained by Brunelli [32].

In the present paper, we propose and study a complex short pulse (CSP) equation

$$q_{xt} + q + \frac{1}{2}\left(|q|^2 q_x\right)_x = 0, \tag{2}$$

and its two-component generalization

$$q_{1,xt} + q_1 + \frac{1}{2}\left((|q_1|^2 + |q_2|^2) q_{1,x}\right)_x = 0, \tag{3}$$

$$q_{2,xt} + q_2 + \frac{1}{2}\left((|q_1|^2 + |q_2|^2) q_{2,x}\right)_x = 0. \tag{4}$$

As will be revealed in the present paper, both the CSP equation and its two-component generalization are integrable guaranteed by the existence of Lax pairs and infinite number of conservation laws. They have N-soliton solutions which can be constructed via Hirota's bilinear method.

The outline of the present paper is organized as follows. In Section 2, we derive the CSP equation and coupled complex short pulse (CCSP) equation from the physical context. In Section 3, by providing the Lax pairs, the integrability of the proposed two equations are confirmed, and further, the conservation laws, both local and nonlocal ones, are investigated. Then N-soliton solutions to both the CSP and CCSP equations are constructed in terms of pfaffians by Hirota's bilinear method in Section 4. In Section 5, soliton-interaction for coupled complex short pulse equation is investigated in details, which shows rich phenomena similar to the coupled nonlinear Schrödinger equation. In particular, they may undergo either elastic or inelastic collision depending on the initial conditions. For inelastic collisions, there is an energy exchange between solitons, which can allow the generation or vanishing of soliton. The dynamics is more richer in compared with the single component case. The paper is concluded by comments and remarks in Section 6.

2. The derivation of the complex short pulse and coupled complex short pulse equations

In this section, following the procedure in [2,7], we derive the complex short pulse equation (2) and its two-component generalization that governs the propagation of ultra short pulse packet along optical fibers.

2.1. The complex short pulse equation

We start with a wave equation for electric field

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\mathbf{E}_{tt} = \mu_0 \mathbf{P}_{tt}, \tag{5}$$

originated from the Maxwell equation. Here $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{P}(\mathbf{r}, t)$ represent the electric field and the induced polarization, respectively, μ_0 is the vacuum permeability, c is the speed of light in vacuum. If we assume the local medium response and only the third-order nonlinear effects governed by $\chi^{(3)}$, the induced polarization consists of two parts, $\mathbf{P}(\mathbf{r}, t) = \mathbf{P}_L(\mathbf{r}, t) + \mathbf{P}_{NL}(\mathbf{r}, t)$, where the linear part

$$\mathbf{P}_L(\mathbf{r}, t) = \varepsilon_0 \int_{-\infty}^{\infty} \chi^{(1)}(t - t') \cdot \mathbf{E}(\mathbf{r}, t')\, dt', \tag{6}$$

and the nonlinear part

$$\mathbf{P}_{NL}(\mathbf{r}, t) = \varepsilon_0 \int_{-\infty}^{\infty} \chi^{(3)}(t - t_1, t - t_2, t - t_3) \times \mathbf{E}(\mathbf{r}, t_1)\mathbf{E}(\mathbf{r}, t_2)\mathbf{E}(\mathbf{r}, t_3)\, dt_1 dt_2 dt_3. \tag{7}$$

Here ε_0 is the vacuum permittivity and $\chi^{(j)}$ is the jth-order susceptibility. Since the nonlinear effects are relatively small in silica fibers, $\mathbf{P}_{NL}$ can be treated as a small perturbation. Therefore, we first consider (5) with $\mathbf{P}_{NL} = 0$. Furthermore, we restrict ourselves to the case that the optical pulse maintains its polarization along the optical fiber, and the transverse diffraction term $\Delta_\perp \mathbf{E}$ can be neglected. In this case, the electric field can be considered to be one-dimensional and expressed as

$$\mathbf{E} = \frac{1}{2}\mathbf{e_1}\left(E(z, t) + c.c.\right), \tag{8}$$

where $\mathbf{e_1}$ is a unit vector in the direction of the polarization, $E(z, t)$ is the complex-valued function, and $c.c.$ stands for the complex conjugate. Conducting a Fourier transform on (5) leads to the Helmholtz equation

$$\tilde{E}_{zz}(z, \omega) + \varepsilon(\omega)\frac{\omega^2}{c^2}\tilde{E}(z, \omega) = 0, \tag{9}$$

where $\tilde{E}(z, \omega)$ is the Fourier transform of $E(z, t)$ defined as

$$\tilde{E}(z, \omega) = \int_{-\infty}^{\infty} E(z, t) e^{i\omega t}\, dt, \tag{10}$$

$\varepsilon(\omega)$ is called the frequency-dependent dielectric constant defined as

$$\varepsilon(\omega) = 1 + \tilde{\chi}^{(1)}(\omega), \tag{11}$$

where $\tilde{\chi}^{(1)}(\omega)$ is the Fourier transform of $\chi^{(1)}(t)$

$$\tilde{\chi}^{(1)}(\omega) = \int_{-\infty}^{\infty} \chi^{(1)}(t) e^{i\omega t}\, dt. \tag{12}$$

Now we proceed to the consideration of the nonlinear effect. Assuming the nonlinear response is instantaneous so that P_{NL} is given by $P_{NL}(z, t) = \varepsilon_0 \varepsilon_{NL} E(z, t)$ [2] where the nonlinear contribution to the dielectric constant is defined as

$$\varepsilon_{NL} = \frac{3}{4}\chi^{(3)}_{xxxx}|E(z, t)|^2. \tag{13}$$

In this case, the Helmholtz equation (9) can be modified as

$$\tilde{E}_{zz}(z, \omega) + \tilde{\varepsilon}(\omega)\frac{\omega^2}{c^2}\tilde{E}(z, \omega) = 0, \tag{14}$$

where

$$\tilde{\varepsilon}(\omega) = 1 + \tilde{\chi}^{(1)}(\omega) + \varepsilon_{NL}. \tag{15}$$

As pointed out in [3,7,8], the Fourier transform $\tilde{\chi}^{(1)}$ can be well approximated by the relation $\tilde{\chi}^{(1)} = \tilde{\chi}^{(1)}_0 - \tilde{\chi}^{(1)}_2 \lambda^2$ if we consider the propagation of optical pulse with the wavelength between 1600 and 3000 nm. It then follows that the linear equation (9) written in Fourier transformed form becomes

$$\tilde{E}_{zz} + \frac{1 + \tilde{\chi}^{(1)}_0}{c^2}\omega^2 \tilde{E} - (2\pi)^2 \tilde{\chi}^{(1)}_2 \tilde{E} + \varepsilon_{NL}\frac{\omega^2}{c^2}\tilde{E} = 0. \tag{16}$$

Applying the inverse Fourier transform to (16) yields a single nonlinear wave equation

$$E_{zz} - \frac{1}{c_1^2}E_{tt} = \frac{1}{c_2^2}E + \frac{3}{4}\chi^{(3)}_{xxxx}\left(|E|^2 E\right)_{tt} = 0. \tag{17}$$