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Direct dynamical energy cascade in the modified KdV equation

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HIGHLIGHTS

- Modulational instability is studied numerically in the mKdV equation framework.
- The D-cascade formation in the Fourier space is highlighted.
- Cascade shape is studied for various values of wave parameters.

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ABSTRACT

In this study we examine the energy transfer mechanism during the nonlinear stage of the Modulational Instability (MI) in the modified Korteweg–de Vries (mKdV) equation. The particularity of this study consists in considering the problem essentially in the Fourier space. A dynamical energy cascade model of this process originally proposed for the focusing NLS-type equations is transposed to the mKdV setting using the existing connections between the KdV-type and NLS-type equations. The main predictions of the *D*-cascade model are outlined and validated by direct numerical simulations of the mKdV equation using the pseudo-spectral methods. The nonlinear stages of the MI evolution are also investigated for the mKdV equation.

1. Introduction

Nonlinear wave systems occur in numerous physical areas from optics to fluid mechanics, from astronomy to geophysics, and one of the most important issues regarding these systems is a description of its energy behavior. One of the most beautiful examples to illustrate this point, is the hypothesis of Kolmogorov on the form of energy spectrum in systems with *strong* turbulence in which the energy spectrum is supposed to have the universal form $\mathcal{E}(\ell) \sim \ell^{-5/3}$ where ℓ is the size of the eddy, [1]. In the kinetic *weak* (or wave) turbulence theory (WTT) dispersive waves play now the role of eddies, and the energy spectrum is again power law $k^{-\alpha}$ where k is the wave length (if dispersion function $\omega \sim k^{\beta}$) and α is not universal any more, [2].

The kinetic WTT is an asymptotic theory which is working for very small nonlinearity $0 < \varepsilon \lessapprox 0.01$, where small parameter ε is usually taken as a product of wave amplitude with wave number, $\varepsilon = Ak$. The smallness of ε is very important while the kinetic WTT is essentially based on the following *assumption*: time scales for 3-, 4-, ..., s-wave resonances are separated and can be studied independently. This assumption breaks at about $\varepsilon \approx 0.1$, *e.g.* [3]. On the other hand, usual laboratory experiments and numerical simulations are performed for $\varepsilon \approx 0.1$–0.4 while for a smaller $\varepsilon \sim 0.01$ corresponding time scales are too long and kinetic energy cascades cannot be observed in an experiment at the present stage of technical facilities, [4,5].

A new model (hereafter referred to as *D*-model) for the formation of the energy spectrum has been developed by E. Kartashova (2012) in [6]; the model can be applied for describing nonlinear wave systems with nonlinearity parameter of the order of $\varepsilon \sim 0.1$–0.4 and wave systems with narrow frequency band excitation. Basic physical mechanism responsible for the formation of the energy spectrum in this model is not a common s-wave resonance but the modulation instability, and the main assumption of the model is that energy cascade is formed by the most unstable modes in the

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system, *i.e.* modes with maximum increment of instability. In [6], the Increment Chain Equation Method (ICEM) was developed for computing dynamical energy spectrum in the systems possessing modulation instability, and applied for the focusing NLS and mNLS, with different levels on nonlinearity, [7,8].

The NLS is a very attractive equation because of its integrability, but unfortunately it gives sometimes not good enough description of the observed physical effects. For instance, modulation instability was discovered in laboratory experiments with water waves and explained by Benjamin & Feir (1967) [9], as instability of a narrow wave packet in the framework of the NLS. However, numerical simulations with NLS demonstrate a symmetric energy cascade in the Fourier space while energy cascade experimentally observed in a water tank, is asymmetric. To cope with this problem, it is necessary to introduce various modifications to NLS, *e.g.* [10,11]. These modifications allow for the realistic values of small parameter, $0 < \varepsilon_{\text{real}} \sim 0.1$–$0.4$, and are more suitable for modeling real physical phenomena.

The Korteweg–de Vries (KdV) equation along with its various modifications is another widely used model equation describing long waves, *i.e.* the region of small wave vectors $kd \ll 1$, d being the mean water depth. This equation does not have the restrictive assumption of narrow spectrum as the NLS equation. However, many modifications of KdV are integrable which is a strong mathematical property. So it is not surprising that the KdV equation has found many applications in different fields of Physics such as the shallow water wave dynamics [12,13], internal waves in two-component fluids [14] and acoustic waves in plasmas [15]. Our motivation to study the KdV-family of equations comes mainly from the numerous real-world applications that it can cover.

Though KdV does not have modulation instability [16], its various modifications do possess this property, under certain conditions. Thus, perturbations of a quasi-periodical wave train with small amplitudes in the generalized KdV equations with nonlinearity of the form $(u^{p+1})_x$

$$\mathrm{gKdV}(u_\pm) \doteq u_t + u_{xxx} + (u^{p+1})_x = 0, \tag{1.1}$$

are modulationally stable if $p < 2$, while they are modulationally unstable if $p > 2$, [17]. A more general version of this result allowing for nonlocal dispersion can be found in [18]. However, all these results do not allow to obtain a nice analytical representation for the instability interval as in [9], and a numerical study is unavoidable.

Another reference point important for our study of mKdV is the following remarkable feature of this equation: it can be reduced, under certain conditions, to the mNLS where the MI can be studied by analytically. This reduction can be made by the variational methods [19] or by standard asymptotical approach as in [14].

Our aim in this paper is to study a particular case with $p = 2$ and one space dimension—the so-called modified Korteweg–de Vries (mKdV) equation

$$\mathrm{mKdV}(u_\pm) \doteq u_t + u_{xxx} \pm 6u^2 u_x = 0, \tag{1.2}$$

with u being a real-valued scalar function, x and t are space and time variables consequently, and the subscripts denote the corresponding partial derivatives. As a starting point for our simulations aiming to study the MI in the mKdV, i.e. $p = 2$ in (1.1), we use the estimates obtained in [17] by combination of analytical results and numerical estimates, namely that for $p = 2$, the wave is spectrally stable for all wave vectors $0 < k^2 < 2$.

It is also shown in [14] that wave packets are unstable only for a positive sign of the coefficient of the cubic nonlinear term in (1.2), and for a high carrier frequency. Being interested in modulation instability, we restrict the study further on the case of focusing mKdV equation:

$$\mathrm{mKdV}(u_+) \doteq u_t + u_{xxx} + 6u^2 u_x = 0. \tag{1.3}$$

In the present paper we aim to study in detail formation and properties of the direct D-cascade in the frame of KdV equation (1.3).

The present manuscript is organized as follows. In Section 2 we give a sketch of a D-cascade formation for this equation and formulate the properties of the cascade and its spectra which should be verified numerically. In Section 3 we describe shortly our numerical approach and present results of our numerical simulations. Finally, the main conclusions of this study are briefly formulated in Section 4.

2. *D*-cascade in the model equation

The main effect of the Modulational Instability (MI) is the disintegration of periodic wavetrains into side bands. Benjamin & Feir (1967) [9] showed that there is a connection between the frequencies, wavenumber and amplitudes of unstable modes in the framework of the focusing (+) Nonlinear Schrödinger (NLS) equation, which reads after a proper re-scaling:

$$\mathrm{NLS}(v_\pm) \doteq \mathrm{i}v_t + v_{xx} \pm |v|^2 v = 0.$$

Namely, they computed the instability interval in the form

$$0 < \Delta\omega \Big/ Ak\omega \leq \sqrt{2}, \tag{2.1}$$

where $\omega(k)$ is the linear dispersion relation, k is the wavenumber and A is the amplitude of the Fourier mode ω. Quantity $\Delta\omega$ is the distance between the parent mode and its side band. It was also shown in [9] that the most unstable mode satisfies the following relation:

$$\Delta\omega \Big/ Ak\omega = 1. \tag{2.2}$$

The use of two assumptions – (a) an energy cascade is formed by the most unstable modes, and (b) the energy fraction p (called cascade intensity) transported from one cascading mode to the next one is constant – allows to construct and to solve an approximate ordinary differential equation for computing amplitudes of cascading modes, [6]. The first constitutive assumption was inspired by the well-known hypothesis of O. Phillips while the second by numerous experimental studies of water waves, *e.g.* [20].

The amplitude of the n-s mode in the cascade can be computed as

$$A(\omega_{\pm n}) = \pm(\sqrt{p} - 1)\int_{\omega_0}^{\omega_{+n}} \frac{\mathrm{d}\omega_n}{\omega_n k_n} + C_\pm(\omega_0, A_0, p). \tag{2.3}$$

Accordingly, by definition the energy $E_n(\omega_n) \propto A^2(\omega_n)$, which provides us with the discrete set of energies of individual harmonics. The spectral density $\mathcal{E}^{(\mathrm{Dir})}(\omega)$ can be now computed:

$$\mathcal{E}^{(\mathrm{Dir})}(\omega) \doteq \lim_{\Delta\omega_n \to 0} \frac{E(\omega_{n+1}) - E(\omega_n)}{\Delta\omega_n}.$$

A similar formula can be written for the inverse cascade as well. However, the limits of integration in (2.3) will be inverted correspondingly to $\int_{\omega_{-n}}^{\omega_0}$.

Besides the form of energy spectra, the D-cascade model allows to make other predictions for the Nonlinear Schrödinger (NLS)-family of the PDEs *e.g.* , the time scale for the D-cascade occurring is $t_{\mathrm{MI}} \propto t/\varepsilon^2$; the distance between two cascading frequencies depends on the steepness of the initial wave train; the cascade can be determined, depending on the choice of excitation parameters ω_0, k_0, A_0; the D-cascade termination can be caused by a few main reasons: stabilization, wave breaking and intermittency. All scenarios were observed experimentally, *e.g.* in [21,22].

Connection between the Nonlinear Schrödinger (NLS) and the mKdV can be established in the following way. As it was shown by