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Barriers to transport and mixing in volume-preserving maps with nonzero flux

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HIGHLIGHTS

- We study area and volume-preserving maps with nonzero net flux.
- Secondary tori are found to play a crucial role in the dynamics of these systems.
- We observe that the escape times of the unbounded orbits are Gamma distributed.
- We provide bounds in parameter space for which secondary tori may exist.
- We describe the asymptotic growth of the resonant regions.

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ABSTRACT

We identify some geometric structures (secondary tori) that restrict transport and prevent mixing in perturbations of integrable volume-preserving systems with nonzero net flux. Unlike the customary KAM tori, secondary tori cannot be continued to the tori present in the integrable system but are generated by resonances and have a contractible direction. We also note that secondary tori persist under the addition of a net flux, which destroys all customary KAM tori.

We introduce a remarkably simple algorithm to analyze the behavior of volume preserving maps and to obtain quantitative properties of the secondary tori. We then implement the algorithm and, after running it, present assertions regarding the distribution of the escape times of the unbounded orbits, the abundance of secondary tori, the size of the resonant regions, and the robustness of the tori under the addition of a mean flux.

1. Introduction

When considering flows in pulsating channels there are two natural questions:

1. Is all of the material flushed out?
2. Is the material thoroughly mixed?

The goal of this paper is to numerically study some geometric structures that prevent a positive answer to these questions.

Volume-preserving maps naturally arise in the study of incompressible fluid flows [1–4] so we will focus our study on these systems. Transport in these maps has been studied [5–12] however this work has concentrated on systems with zero net flux. In this case (under very mild non-degeneracy conditions) codimension one, non-contractible tori foliate a set of positive measure of the phase space of the nearly integrable system and act as boundaries to transport. This set is topologically a Cantor set times a torus.

The results of the above studies do not directly apply to the problem of channel flows since these systems exhibit nonzero net flux. Indeed, as is well known, non-contractible tori cannot exist when there is a non-zero mean flux. We discover that the main objects that prevent transport in positive flux volume-preserving maps are invariant tori of codimension one which cannot be continued to the integrable system and have contractible directions. These tori, which we will refer to as *secondary*, are generated by resonances.

We point out that systems with non-trivial flux appear naturally in several physical systems (e.g. in flows in channels when there is an average flow). Of course, these systems have been staples in fluid mechanics engineering where behavior in pipes is of paramount importance. The literature considering invariant manifolds and other objects from dynamical systems seems to be more

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limited [13–16]. The paper [17] shows that invariant structures in fluids can serve as places where bacteria can attach and form biofilms. We have not looked at the biological or chemical implications of secondary tori for reacting fluids or for biologically active ones, but it seems that confinement can lead to interesting physical effects.

In this paper we use an exceedingly simple method to compute secondary tori. We generate random initial conditions and determine if the resulting orbits remain bounded. We observe that the orbits that remain bounded are in regions bounded by secondary tori. These regions contain other secondary tori as well as chaotic regions. We also compute several quantitative properties of the orbits.

1.1. Preliminaries

We consider measure-preserving mappings of $\mathbb{T}^d \times \mathbb{R}$ and focus on the most important physical cases which have $d = 1, 2$. A simple example is

$$(x', z') = f_\lambda(x, z) = (x + \Omega(z'), z + \lambda) \tag{1}$$

with angles $x \in \mathbb{T}^d$ and action $z \in \mathbb{R}$. Note that λ can be interpreted as a mean flux. If $\lambda \neq 0$, all the trajectories are unbounded and escape, however the system remains completely unmixed. On the other hand, if $\lambda = 0$, the flow is stratified. Every orbit lies on an invariant torus $\mathcal{T}_z = \mathbb{T}^d \times \{z_0\}$ with constant action. All tori homotopic to $\mathcal{T}_z$ are referred to as *rotational*. Note that rotational tori separate the phase space into two regions which are bounded in the z variable. If a rotational torus is invariant, one orbit cannot cross from one side to another. Hence, if a system has invariant rotational tori, the orbits have bounded z.

A system with richer dynamics than the integrable case is the model proposed in [5]

$$(x', z') = f_{\varepsilon,\lambda}(x, z) = (z + \Omega(z'), z + \varepsilon g(x) + \lambda) \tag{2}$$

where $g(x)$ is an average-zero function. This map was derived as a model for the typical behavior of a volume-preserving system near resonance [5] and therefore provides a canonical example for study. For simplicity, we assume Ω and g are analytic. When $\lambda = 0$ there are KAM results [6–8,18] that show that when $|\varepsilon| \ll 1$ the model (2) possesses a set of invariant rotational tori of positive measure on which the dynamics is conjugate to a rigid rotation. Orbits cannot cross these tori, hence all orbits remain bounded. Moreover, they also prevent complete mixing since they separate regions of phase space.

When $\lambda \neq 0$ rotational tori cannot exist regardless of the size of ε. As is well known, the net flux is defined to be the volume of the region bounded between a rotational torus and its iterate (giving a sign to the regions above and the opposite sign to the regions below). It is an easy consequence of volume preservation that the net flux is independent of the torus considered [19]. This is similar to the fact that we can measure the flow of a stream by intercepting it with any surface of section. Hence, if there is an invariant rotational torus, we obtain that the net flux has to be zero by computing the net flux based on this invariant torus. Conversely, if the net flux is not zero, there cannot be any rotational invariant tori.

Nevertheless, as we will see, for $0 < |\varepsilon| \ll 1$ the perturbations generate some new d-dimensional invariant structures that are not present in the integrable case and are contractible to lower dimensional tori. These structures persist when the system is perturbed in both ε and λ, as can be established using KAM theory. Since they are codimension one they separate regions of space. Any orbits trapped within the torus cannot escape and therefore cannot become unbounded.

Secondary tori in twist maps, often referred to as *islands*, have been extensively studied – see, for example, [20–22] – and have been shown to exist for arbitrarily large perturbations [23,24]. It has also been argued that they are the dominant structure preventing equidistribution in coupled map lattices [25]. In higher dimensional maps the secondary tori appear as *tubes* [9]. It is an open question as to whether these tubes also exist when the perturbation is very strong. A forthcoming paper by the authors [26] establishes rigorously the existence of these tubes in near-integrable volume-preserving maps under some mild and verifiable non-degeneracy conditions.

We will explore the effect of these secondary tori by adding a flux term to two oft-studied systems. The Standard Map,

$$\begin{aligned} x' &= x + z' \\ z' &= z - \frac{\varepsilon}{2\pi}\sin(2\pi x) + \lambda, \end{aligned} \tag{3}$$

is an example of (2) with $d = 1$. It was introduced by Chirikov [27] as a model for the behavior of a generic Hamiltonian system near resonance, however has also been used to model many natural phenomena, see [28] for a comprehensive overview. The Standard Volume Preserving Map, [5]

$$\begin{aligned} x_1' &= x_1 + \gamma + z' \\ x_2' &= x_2 - \delta + \beta(z')^2 \\ z' &= z + \varepsilon(\sin(2\pi x_1) + \sin(2\pi x_2) + \sin(2\pi(x_1 - x_2))) + \lambda, \end{aligned} \tag{4}$$

is of the form of (2) with $d = 2$. Following [10–12] we use the parameters $\beta = 2$ and $\gamma = \frac{1}{2}(-1 + \sqrt{5})$ and set $\delta = 0$. Maps of this form naturally arise in the study of incompressible fluid flows, see, for example, [1–3]. The addition of the flux term λ allows us to model these fluids moving through a channel, such as water in a pipe or blood through an artery.

In Section 2 we outline numerical techniques to analyze the dynamics of (3) and (4). The main observation is that a non-zero flux destroys all rotational invariant tori. However, this flux does not destroy the secondary tori which bound regions of phase space.

The behavior of the unbounded orbits, those not enclosed by secondary tori, is described in Section 3. We provide evidence for the conjecture that the escape time of the orbits can be modeled as a Gamma random variable whenever secondary tori are present.

In Section 4 we describe the bounded orbits and the secondary tori that enclose them. In particular we study the abundance of the tori in parameter space and the Lyapunov exponents and global rotation vectors of the trapped orbits. We will also present a conjecture on the region in parameter space for which secondary tori may exist and provide numerical estimates for the size of the resonant regions.

2. Numerical methods

The numerical methods we employ to study these maps are exceptionally simple. We select random initial conditions uniformly on $0 \leq x, z \leq 1$ and iterate the map $f_{\varepsilon,\lambda}$ a specified number of times, $n_{\max}$, to generate an orbit. We classify an orbit as unbounded if the action z grows larger than some predetermined threshold z_{th}, i.e. if $|z| > z_{th}$. Otherwise, the orbit is said to be bounded. Both constants z_{th} and $n_{\max}$ must be chosen, but we note that the results do not change once obvious pitfalls, such as choosing z_{th} or $n_{\max}$ significantly too small, are avoided (we will quantify this in Section 3).

If the orbit is bounded we compute the maximal Lyapunov exponent

$$\mathcal{L} = \lim_{n\to\infty} \log_{10} \frac{1}{n} \|Df^n(x_n, z_n)\vec{v}\|, \tag{5}$$