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Phase description of oscillatory convection with a spatially translational mode

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HIGHLIGHTS

- We develop a phase reduction theory for oscillatory convection with a spatial mode.
- The theory can be considered as a phase description method for limit-torus solutions.
- We derive phase sensitivity functions for spatial and temporal phases of convection.
- We can quantify spatiotemporal phase responses of convection to weak perturbations.
- We can analyze spatiotemporal phase synchronization between weakly coupled systems.

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ABSTRACT

We formulate a theory for the phase description of oscillatory convection in a cylindrical Hele–Shaw cell that is laterally periodic. This system possesses spatial translational symmetry in the lateral direction owing to the cylindrical shape as well as temporal translational symmetry. Oscillatory convection in this system is described by a limit-torus solution that possesses two phase modes; one is a spatial phase and the other is a temporal phase. The spatial and temporal phases indicate the "position" and "oscillation" of the convection, respectively. The theory developed in this paper can be considered as a phase reduction method for limit-torus solutions in infinite-dimensional dynamical systems, namely, limit-torus solutions to partial differential equations representing oscillatory convection with a spatially translational mode. We derive the phase sensitivity functions for spatial and temporal phases; these functions quantify the phase responses of the oscillatory convection to weak perturbations applied at each spatial point. Using the phase sensitivity functions, we characterize the spatiotemporal phase responses of oscillatory convection to weak spatial stimuli and analyze the spatiotemporal phase synchronization between weakly coupled systems of oscillatory convection.

1. Introduction

Nature provides abundant examples of rhythmic systems and synchronization phenomena [1–5]. Each rhythmic system is typically described by an ordinary differential equation that possesses a limit-cycle solution. The phase reduction method for ordinary limit-cycle oscillators has been well established and successfully applied to analyze the synchronization properties of the oscillators [1–3,6–10]. There also exist rhythmic spatiotemporal patterns described by limit-cycle solutions to partial differential equations [11–18].

We recently developed a phase description method for limit-cycle solutions to the following partial differential equations: the nonlinear Fokker–Planck equations that represent the collective dynamics of globally coupled noisy dynamical elements [19], the fluid equations that represent the dynamics of the temperature field in ordinary Hele–Shaw cells [20], and the reaction–diffusion equations that represent rhythmic spatiotemporal patterns in chemical and biological systems [21]. However, there are also examples of spatiotemporal rhythms in systems that further possess spatial translational symmetry; these spatiotemporal rhythms cannot be described by limit-cycle solutions.

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For example, rotating annuli and spheres possess continuously rotational symmetry, i.e., continuously translational symmetry in the rotating direction [17,18,22,23]. Consequently, the emergence of spatiotemporal rhythms in such systems brings up two phase modes, i.e., a spatial phase and a temporal phase. Such spatiotemporal rhythms are described by limit-torus solutions. Synchronization of spatiotemporal rhythms with two phase modes has been experimentally investigated using systems of rotating fluid annuli that exhibit traveling and oscillating convection (i.e., amplitude vacillation [18]), which is analogous to atmospheric circulation [24,25]. Therefore, a phase description method for limit-torus solutions to partial differential equations is desirable.

In this paper, as the first step, we consider oscillatory convection in a cylindrical Hele–Shaw cell that is laterally periodic. An ordinary Hele–Shaw cell is a rectangular cavity where the gap between two vertical walls is much smaller than the extent of the other two spatial dimensions, and the fluid in the cavity exhibits oscillatory convection under the appropriate parameter conditions (see Refs. [26,27] and references therein). The cylindrical Hele–Shaw cell is a cylindrical version of the ordinary Hele–Shaw cell that possesses spatial translational symmetry in the lateral direction owing to the cylindrical shape. Oscillatory convection in the cylindrical Hele–Shaw cell is therefore described by a limit-torus solution that possesses both spatial and temporal phases.

Here, we formulate a theory for the phase description of oscillatory convection in the cylindrical Hele–Shaw cell. The theory can be considered as a phase reduction method for limit-torus solutions to partial differential equations. The theory can also be considered as a generalization of our phase description method for limit-cycle solutions to partial differential equations such as the nonlinear Fokker–Planck equations [19], fluid equations [20], and reaction–diffusion equations [21]. The phase reduction method for limit-torus solutions enables us to describe the dynamics of the oscillatory convection by two phases (i.e., spatial and temporal phases), and facilitates theoretical analysis of the spatiotemporal phase synchronization properties of the oscillatory convection. On the basis of phase reduction, we characterize the spatiotemporal phase responses of oscillatory convection to weak impulses and analyze the spatiotemporal phase synchronization between weakly coupled systems exhibiting oscillatory convection.

This paper is organized as follows. In Section 2, we formulate a theory for the phase description of oscillatory convection with a spatially translational mode; supplemental information of the theory is given in Appendices A and B. In Section 3, we illustrate the theory using a numerical analysis of the oscillatory convection. In Section 4, we make a comparison between the theory and direct numerical simulations. Concluding remarks are given in Section 5.

2. Phase description of oscillatory convection

In this section, we formulate a theory for the phase description of oscillatory convection in a cylindrical Hele–Shaw cell that is laterally periodic. The theory can be considered as an extension of our phase description method for oscillatory convection in the ordinary Hele–Shaw cell [20] to that in the cylindrical Hele–Shaw cell.

2.1. Dimensionless form of the governing equations

The dynamics of the temperature field $T(x, y, t)$ in the cylindrical Hele–Shaw cell is described by the following dimensionless form (see Ref. [26] and references therein):

$$\frac{\partial}{\partial t} T(x, y, t) = \nabla^2 T + J(\psi, T). \tag{1}$$

The Laplacian and Jacobian are respectively given by

$$\nabla^2 T = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) T, \tag{2}$$

$$J(\psi, T) = \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial T}{\partial x}, \tag{3}$$

where we assumed that the curvature effects due to the cylindrical shape are negligible (see Refs. [28,29] for curvature effects, although the subject of these references is not thermal convection but viscous fingering). The first and second terms on the right-hand side of Eq. (1) represent diffusion and advection, respectively. The stream function $\psi(x, y, t)$ is determined from the temperature field $T(x, y, t)$ as follows:

$$\nabla^2 \psi(x, y, t) = -\mathrm{Ra} \frac{\partial T}{\partial x}, \tag{4}$$

where the Rayleigh number is denoted by Ra. The stream function $\psi(x, y, t)$ also gives the fluid velocity field $\boldsymbol{v}(x, y, t)$, i.e.,

$$\boldsymbol{v}(x, y, t) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right). \tag{5}$$

Fig. 1 shows a schematic diagram of the cylindrical Hele–Shaw cell. The system is defined in the following rectangular region: $x \in [0, 2]$ and $y \in [0, 1]$. Because this Hele–Shaw cell has a cylindrical shape, the system possesses a 2-periodicity with respect to x. The boundary conditions for the temperature field $T(x, y, t)$ are given by

$$T(x + 2, y, t) = T(x, y, t), \tag{6}$$

$$T(x, y, t)\Big|_{y=0} = 1, \qquad T(x, y, t)\Big|_{y=1} = 0, \tag{7}$$

where the temperature at the bottom ($y = 0$) is higher than that at the top ($y = 1$). The stream function $\psi(x, y, t)$ satisfies the periodic boundary condition on x and the Dirichlet zero boundary condition on y, i.e.,

$$\psi(x + 2, y, t) = \psi(x, y, t), \tag{8}$$

$$\psi(x, y, t)\Big|_{y=0} = \psi(x, y, t)\Big|_{y=1} = 0. \tag{9}$$

Owing to the homogeneity of Eqs. (1) (4) and the periodic boundary condition on x, given in Eqs. (6) (8), this system possesses continuous spatial translational symmetry with respect to x. We also note that no conserved quantity exists in this system.

There also exists spatial reflection symmetry in this system, i.e., Eqs. (1) and (4) are invariant under the simultaneous transformation of $(x, y) \to (-x, y)$ and $(\psi, T) \to (-\psi, T)$; however, the theory formulated below does not require this reflection symmetry.

2.2. Convective components of the temperature field

To simplify the boundary conditions in Eq. (7), we consider the convective component $X(x, y, t)$ of the temperature field $T(x, y, t)$ as follows:

$$T(x, y, t) = (1 - y) + X(x, y, t). \tag{10}$$

Substituting Eq. (10) into Eqs. (1) and (4), we derive the following equations:

$$\frac{\partial}{\partial t} X(x, y, t) = \nabla^2 X + J(\psi, X) - \frac{\partial \psi}{\partial x}, \tag{11}$$

and

$$\nabla^2 \psi(x, y, t) = -\mathrm{Ra} \frac{\partial X}{\partial x}. \tag{12}$$