Physica D 295–296 (2015) 103–121

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Riemann–Hilbert approach to gap probabilities for the Bessel process

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HIGHLIGHTS

- Gap probabilities for the single and multi-time Bessel process are considered.
- Fredholm determinants of such processes are related to Riemann–Hilbert problems.
- In the single-time case the Fredholm determinant is related to Painlevé 3 equation.

ARTICLE INFO

Article history:
Received 7 September 2013
Received in revised form
10 December 2014
Accepted 23 December 2014
Available online 1 January 2015
Communicated by P.D. Miller

Keywords:
Determinantal point processes
Riemann–Hilbert problem
Integrable kernel

ABSTRACT

We consider the gap probability for the Bessel process in the single-time and multi-time case. We prove that the scalar and matrix Fredholm determinants of such process can be expressed in terms of determinants of integrable kernels in the sense of Its–Izergin–Korepin–Slavnov and thus related to suitable Riemann–Hilbert problems. In the single-time case, we construct a Lax pair formalism and we derive a Painlevé III equation related to the Fredholm determinant.

1. Introduction

Many models in Mathematical Physics rely on the notion of a determinantal random point process (DPP). A few examples are offered by the statistical distribution of the eigenvalues of random matrix models pioneered by Dyson [1], certain models of random growth of crystals [2–4], and mutually avoiding random walkers (usually referred as Dyson's processes).

The present paper falls into the last category and establishes a connection between certain "gap" probabilities and a particular class of boundary value problems in the complex plane, generally referred to as "Riemann–Hilbert problems" (see e.g. [5]). We will show that such boundary value problem, suitably formulated, allows to express the "gap" probabilities in terms of theory of equation of Painlevé type; this relationship is quite well-known originally in two dimensional statistical physics [6] and it was extensively studied in the eighties and nineties [7–12].

In order to frame our results in a narrower context, we mention the well-known "Tracy–Widom" distribution [11] which describes the fluctuations of the largest eigenvalue of a random Gaussian matrix (suitably scaled) in terms of the solution of a nonlinear ODE (Painlevé II). Similarly in [12] the authors connected the fluctuation of the smallest eigenvalue of the "Laguerre ensemble" to the third member of the Painlevé hierarchy. Our results are closely related to these and will extend this connection to the case of the "Bessel process" using a completely different method.

We also mention the recent paper [13] where a new Lax pair related to the Painlevé III equation is defined and the first component of their eigenvector is the smallest eigenvalue probability near the hard edge in the large n limit of the Laguerre ensemble.

Before getting into the details pertinent to the Bessel process we will briefly review for the sake of the reader the main concepts about DPP in timeless and dynamic regimes and explain the connection with our case. For surveys on DPP, we refer to Soshnikov [14] and Johansson [15].

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http://dx.doi.org/10.1016/j.physd.2014.12.009
0167-2789/

Determinantal point process. Consider a random collection of points on the real line. A configuration $\mathcal{X}$ is a subset of $\mathbb{R}$ that locally contains a finite number of points, i.e. $\#(\mathcal{X} \cap [a,b]) < +\infty$ for every bounded interval $[a,b] \subset \mathbb{R}$. Then, a (locally finite) point process on $\mathbb{R}$ is a probability measure on the space of all configurations, so that, loosely speaking, it is possible to evaluate the probability of any given configuration.

Given a point process on $\mathbb{R}$, the mapping $A \mapsto \mathbb{E}[\#(\mathcal{X} \cap A)]$, which assigns to a Borel set A the expected value of the number of points in A, is a measure on $\mathbb{R}$. We assume there exists a density ρ_1 with respect to Lebesgue measure and we call it 1-point correlation function for the point process. Then, we have

$$\mathbb{E}[\#(\mathcal{X} \cap A)] = \int_A \rho_1(x)\mathrm{d}x \tag{1.1}$$

and $\rho_1(x)dx$ represents the probability to have a point in the infinitesimal interval $[x, x+dx]$. In general, given disjoints sets $A_1, \ldots, A_k$, we have

$$\mathbb{E}\left[\prod_{j=1}^{k} \#(\mathcal{X} \cap A_j)\right] = \int_{A_1} \cdots \int_{A_k} \rho_k(x_1, \ldots, x_k)\mathrm{d}x_1 \ldots \mathrm{d}x_k \tag{1.2}$$

i.e. the expected number of k-tuples $(x_1, \ldots, x_k) \in \mathcal{X}^k$ such that $x_j \in A_j$ for every j. In case the A_j's are not disjoint it is still possible to define the quantity above, with little modifications.

A point process with correlation functions $\{\rho_k\}_{k\geq 1}$ is a DPP if there exists a kernel $K(x,y)$ such that for every k and every $x_1, \ldots, x_k$ we have

$$\rho_k(x_1, \ldots, x_k) = \det[K(x_i, x_j)]_{i,j=1}^{k}. \tag{1.3}$$

In a DPP all quantities of interest can be expressed in terms of K. In particular, given a Borel set A, we are interested in the gap probability, i.e. the probability to find no points in A. It is possible to show that such quantity is equal to the Fredholm determinant

$$\det(\mathrm{Id} - \chi_A K \chi_A) \tag{1.4}$$

of the (trace class) integral operator K defined by

$$K[f](x) = \int_{\mathbb{R}} K(x,y) f(y)\mathrm{d}y \tag{1.5}$$

and restricted to the Borel set A (χ_A is the projection on such subset).

An example of a DPP is the set of eigenvalues of a random matrix ensemble as in the case of the aforementioned Gaussian and Laguerre ensembles. The theory of gap probabilities for DPP has been extensively studied (see [16–19] to mention a few).

Remark 1.1. The same arguments are valid when performing a scaling limit of the process, as the number of points goes to ∞, at different sectors in the domain of the points (see for example [14]). As we will see, this is the case of the Bessel process, which appears as limiting kernel of certain matrix ensembles as the dimension of the matrix goes to infinity.

A generalization of such theory is given by introducing multiple times $0 < \tau_1 < \cdots < \tau_n$ at which one can simultaneously study the behaviour of the points. We call a time-dependent process "multi-time process".

The idea behind the introduction of multi-time processes is to be able to consider not only static models in timeless equilibrium, but also dynamical systems which may be in an arbitrary non-equilibrium state changing with time.

The first implementation of this dynamic regime was proposed by Dyson [1] for the study of the random eigenvalues of a Gaussian ensemble.

Given a collection of times $\{\tau_k\}_{k=1,\ldots,n}$, within a fixed time interval, say $(0, T)$, and subsets $A_k \subset \mathbb{R},\ k = 1, \ldots, n$, the quantity of interest is the probability that for all k no points lie in A_k at time τ_k. We call again this quantity "gap probability".

Applying classical results from Karlin and McGregor [20] and Eynard and Mehta [21], it can be proved that the gap probability is equal to the Fredholm determinant of a suitable integral operator $[K]$, with matrix kernel $[K_{ij}]_{i,j=1}^{n}$, restricted to the sets $\vec{A} = A_1 \sqcup \cdots \sqcup A_n$:

$$P\,(\text{no points in } A_k \text{ at time } \tau_k,\ \forall k) = \det(\mathrm{Id} - \chi_{\vec{A}}\,[K]\,\chi_{\vec{A}}). \tag{1.6}$$

The Bessel process. We introduce now the subject of the present paper. The Bessel process is a determinantal point process as detailed above [14] defined in terms of a trace-class integral operator acting on $L^2(\mathbb{R}_+)$, with kernel

$$K_B(x,y) = \frac{J_\nu(\sqrt{x})\sqrt{y}J_{\nu+1}(\sqrt{y}) - J_{\nu+1}(\sqrt{x})\sqrt{x}J_\nu(\sqrt{y})}{2(x-y)} \tag{1.7}$$

where J_ν are Bessel functions with parameter $\nu > -1$.

The Bessel kernel K_B arose originally as the correlation function in the scaling limit of the Laguerre and Jacobi Unitary Ensembles near the hard edge of their spectrum at zero [22–24] as well as of generalized LUEs and JUEs [25,26].

In this article we focus on the gap probabilities of this process. In particular, we will be concerned with the Fredholm determinant of such operator on a collection of (finite) intervals $I := \bigcup_{i=1}^{N}[a_{2i-1}, a_{2i}]$, i.e. the Tracy–Widom distribution $\det(\mathrm{Id} - K_B\chi_I)$, and the emphasis is on the determinant thought of as a function of the endpoint $a_i,\ i = 1, \ldots, 2N$.