Physica D 292–293 (2015) 29–45

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Coleman–Gurtin type equations with dynamic boundary conditions

Ciprian G. Gal [a], Joseph L. Shomberg [b,*]

[a] Department of Mathematics, Florida International University, Miami, FL 33199, USA

[b] Department of Mathematics and Computer Science, Providence College, Providence, RI 02918, USA

HIGHLIGHTS

- We consider boundary conditions with temperature dependent sources/sinks and memory.
- We consider memory functions on the boundary and in the interior that differ.
- We consider nonlinear terms satisfying nonlinear balance conditions.
- We develop a general framework allowing for both weak and smooth initial data.
- We extend a Galerkin approximation scheme.

ARTICLE INFO

Article history:
Received 26 April 2014
Received in revised form
26 June 2014
Accepted 22 October 2014
Available online 31 October 2014
Communicated by R. Temam

Keywords:
Coleman–Gurtin equation
Dynamic boundary conditions with memory
Heat conduction
Heat equations

ABSTRACT

We present a new formulation and generalization of the classical theory of heat conduction with or without fading memory. As a special case, we investigate the well-posedness of systems which consist of Coleman–Gurtin type equations subject to dynamic boundary conditions, also with memory. Nonlinear terms are defined on the interior of the domain and on the boundary and subject to either classical dissipation assumptions, or to a nonlinear balance condition in the sense of Gal (2012). Additionally, we do not assume that the interior and the boundary share the same memory kernel.

1. Introduction

In recent years there has been an explosive growth in theoretical results concerning dissipative infinite-dimensional systems with memory including models arising in the theory of heat conduction in special materials and the theory of phase-transitions. The mathematical and physical literature, concerned primarily with qualitative/quantitative properties of solutions to these models, is quite extensive and much of the work before 2002 is largely referenced in the survey paper by Grasselli and Pata [1]. More recent results and updates can be found in [2–5] (cf. also [6,7]). A basic evolution equation considered in these references is that for a homogeneous and isotropic heat conductor occupying a d-dimensional (bounded) domain Ω with sufficiently smooth boundary $\Gamma = \partial\Omega$ and reads

$$\partial_t u - \omega\Delta u - (1-\omega)\int_0^\infty k_\Omega(s)\,\Delta u(x,t-s)\,ds + f(u) = 0, \tag{1.1}$$

in $\Omega \times (0,\infty)$. Here $u = u(t)$ is the (absolute) temperature distribution, $\omega > 0$, $r = -f(u(t))$ is a temperature dependent heat supply, and $k_\Omega : [0,\infty) \to \mathbb{R}$ is a continuous nonnegative function, smooth on $(0,\infty)$ and vanishing at infinity, and summable. As usual, (1.1) is derived by assuming the following energy balance equation

$$\partial_t e + \operatorname{div}(q) = r$$

by considering the following relationships:

$$e = e_\infty + c_0 u,$$
$$q = -\omega\nabla u - (1-\omega)\int_0^\infty k_\Omega(s)\,\nabla u(x,t-s)\,ds, \tag{1.2}$$

* Corresponding author.
E-mail addresses: cgal@fiu.edu (C.G. Gal), jshomber@providence.edu (J.L. Shomberg).

http://dx.doi.org/10.1016/j.physd.2014.10.008
0167-2789/

for some constants $e_\infty, c_0 > 0$. Eq. (1.1) is always subject to either homogeneous Dirichlet ($u = 0$) or Neumann boundary conditions ($\partial_n u = 0$) on $\Gamma \times (0, \infty)$. The first one asserts that the temperature is kept constant and close to a given reference temperature at Γ for all time $t > 0$, while the second "roughly" states that the system is thermally isolated from outside interference. This equation is also usually supplemented by the "initial" condition $\widetilde{u} : (-\infty, 0] \to \mathbb{R}$ such that

$$u_{|t\in(-\infty,0]} = \widetilde{u} \quad \text{in } \Omega. \tag{1.3}$$

These choices of boundary conditions, although help simplify substantially the mathematical analysis of (1.1)–(1.3), are actually debatable in practice since in many such systems it is usually difficult, if not impossible, to keep the temperature constant at Γ for all positive times without exerting some additional kind of control at Γ for $t > 0$. A matter of principle also arises for thermally isolated systems in which, in fact, the correct physical boundary condition for (1.1) turns out to be the following

$$\begin{aligned} q \cdot n &= \omega \partial_n u + (1-\omega) \int_0^\infty k_\Omega (s)\, \partial_n u (x, t-s)\, ds \\ &= 0 \quad \text{on } \Gamma \times (0, \infty), \end{aligned} \tag{1.4}$$

see, for instance, [8, Section 6]. Indeed, the condition $\partial_n u = 0$ on $\Gamma \times (0, \infty)$ implies (1.4), say when u is a sufficiently smooth solution of (1.1)–(1.3), but clearly the converse cannot hold in general.

In the classical theory of heat conduction, it is common to model a wide range of diffusive phenomena including heat propagation in homogeneous isotropic conductors, but generally it is assumed, as above, that surface (i.e., boundary) conditions are completely static or stationary. In some important cases this perspective neglects the contribution of boundary sources to the total heat content of the conductor. A first step to remedy this situation was done in Goldstein [9] for heat equations. The approach presented there introduces dynamic boundary conditions into an *ad hoc* fashion and lacks some rigour in the case of reaction–diffusion equations. In the final section of the paper we will make use of the usual physical principles and present a new formulation and generalization of the classical theory. Our general approach follows that of Coleman and Mizel [8] which regards the second law of thermodynamics as included among the laws of physics and which is compatible with the principle of equipresence in the sense of Truesdell and Toupin (see the Appendix). Thus, this new formulation is expected to give a solid foundation to the arguments employed in derivations of the heat equation with "dynamic" boundary conditions developed in Goldstein [9], or in models for phase transitions developed in Gal and Grasselli [10,11]. Accounting for the presence of boundary sources, the new formulation naturally leads to dynamic boundary conditions for the temperature function u and that contain the above static conditions (especially, (1.4)) as special cases (see the Appendix). In particular, we derive on $\Gamma \times (0, \infty)$, the following boundary condition for (1.1):

$$\begin{aligned} &\partial_t u - \nu \Delta_\Gamma u + \omega \partial_n u + g(u) + (1-\omega) \\ &\quad \times \int_0^\infty k_\Omega (s)\, \partial_n u (x, t-s)\, ds + (1-\nu) \\ &\quad \times \int_0^\infty k_\Gamma (s) (-\Delta_\Gamma + \beta) u (x, t-s)\, ds = 0, \end{aligned} \tag{1.5}$$

for some $\nu \in (0, 1)$ and $\beta > 0$. Here $k_\Gamma : [0, \infty) \to \mathbb{R}$ is also a smooth nonnegative, summable function over $(0, \infty)$ such that k_Γ is vanishing at infinity. The last two boundary terms on the left-hand side of (1.5) are due to contributions coming from a (linear) heat exchange rate between the bulk Ω and the boundary Γ, and boundary fluxes, respectively (cf. the Appendix).

Our goal in this paper is to extend the previous well-posedness results of [2–5,1,6,7,12–14] in the following directions:

- by allowing general boundary processes take place also on Γ, Eq. (1.1) is now subject to boundary conditions of the form (1.5);
- we consider more general functions $f, g \in C^1(\mathbb{R})$ satisfying either classical dissipation assumptions, or more generally, nonlinear balance conditions allowing for bad behaviour of f, g at infinity;
- we develop a general framework allowing for both weak and smooth initial data for (1.1), (1.5), and possibly *different* memory functions k_Ω, k_Γ.
- we extend a Galerkin approximation scheme whose explicit construction is crucial for the existence of strong solutions.

The paper is organized as follows. In Section 2, we provide the functional setup. In Section 3, we prove theorems concerning the well-posedness of the system, based on (1.1), (1.5), generated by the new formulation. In the final section, we present a rigorous formulation and examples in which (1.5) naturally occurs for (1.1).

2. Past history formulation and functional setup

As in [3] (cf. also [1]), we can introduce the so-called integrated past history of u, i.e., the auxiliary variable

$$\eta^t (x, s) = \int_0^s u (x, t-y)\, dy,$$

for $s, t > 0$. Setting

$$\begin{aligned} \mu_\Omega (s) &= -\omega^{-1} (1-\omega)\, m'_\Omega (s), \\ \mu_\Gamma (s) &= -\nu^{-1} (1-\nu)\, m'_\Gamma (s), \end{aligned} \tag{2.1}$$

assuming that m_S, $S \in \{\Omega, \Gamma\}$, is sufficiently smooth and vanishing at ∞, formal integration by parts into (A.28)–(A.29) yields

$$\begin{aligned} &(1-\omega) \int_0^\infty m_\Omega (s)\, \Delta u (x, t-s)\, ds \\ &\quad = \omega \int_0^\infty \mu_\Omega (s)\, \Delta \eta^t (x, s)\, ds, \\ &(1-\omega) \int_0^\infty m_\Omega (s)\, \partial_n u (x, t-s)\, ds \\ &\quad = \omega \int_0^\infty \mu_\Omega (s)\, \partial_n \eta^t (x, s)\, ds \end{aligned}$$

and

$$\begin{aligned} &(1-\nu) \int_0^\infty m_\Gamma (s) (-\Delta_\Gamma u (t-s) + \beta u (t-s))\, ds \\ &\quad = \nu \int_0^\infty \mu_\Gamma (s) \left(-\Delta_\Gamma \eta^t (s) + \beta \eta^t (s)\right) ds. \end{aligned} \tag{2.2}$$

Thus, we consider the following formulation.

Problem P. Find a function $\left(u, \eta^t\right)$ such that

$$\begin{aligned} &\partial_t u - \omega \Delta u - \omega \int_0^\infty \mu_\Omega (s)\, \Delta \eta^t (s)\, ds \\ &\quad + \alpha \omega \int_0^\infty \mu_\Omega (s)\, \eta^t (x, s)\, ds + f(u) = 0, \end{aligned} \tag{2.3}$$

in $\Omega \times (0, \infty)$,

$$\begin{aligned} &\partial_t u - \nu \Delta_\Gamma u + \omega \partial_n u + \omega \int_0^\infty \mu_\Omega (s)\, \partial_n \eta^t (s)\, ds \\ &\quad + \nu \int_0^\infty \mu_\Gamma (s) \left(-\Delta_\Gamma \eta^t (s) + \beta \eta^t (s)\right) ds + g(u) = 0, \end{aligned} \tag{2.4}$$

on $\Gamma \times (0, \infty)$, and

$$\partial_t \eta^t (s) + \partial_s \eta^t (s) = u(t), \quad \text{in } \overline{\Omega} \times (0, \infty), \tag{2.5}$$