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Premixed-flame shapes and polynomials

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HIGHLIGHTS

- A nonlinear nonlocal equation for isolated crests of unstable gaseous flames is studied.
- Polynomials encoding the 2N flame-slope poles nearly follow a 3-term recurrence.
- Meixner–Pollaczek polynomials soon provide accurate crest shapes as N grows.
- Discretized equations for squeezed Burgers crests also lead to those polynomials.
- Despite similarities such an approximation is still missing for periodic patterns.

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ABSTRACT

The nonlinear nonlocal Michelson–Sivashinsky equation for isolated crests of unstable flames is studied, using pole-decompositions as starting point. Polynomials encoding the numerically computed 2N flame-slope poles, and auxiliary ones, are found to closely follow a Meixner–Pollaczek recurrence; accurate steady crest shapes ensue for $N \geq 3$. Squeezed crests ruled by a discretized Burgers equation involve the same polynomials. Such explicit approximate shapes still lack for finite-N pole-decomposed periodic flames, despite another empirical recurrence.

1. Introduction

Sivashinsky [1] derived the first nonlinear evolution equation for the amplitude $\phi(t, x)$ of wrinkling of premixed gaseous flames subject to the nonlocal Darrieus [2]–Landau [3] (DL) hydrodynamic instability, when the Atwood number $\mathcal{A} = (E-1)/(E+1)$ based on the fresh-to-burnt density ratio $E > 1$ is small. The evolution equation, first studied numerically in [4], is:

$$\phi_t + \frac{1}{2}\phi_x^2 - \nu\phi_{xx} + \mathcal{H}\{\phi_x\} = 0 \tag{1}$$

once put in scaled form, and is often termed the Michelson–Sivashinsky (MS) equation. The subscripts in there denote partial derivatives in scaled time (t) and abscissa (x). The constant Markstein (dimensionless-) 'length' $\nu > 0$ controls the curvature-induced $\sim\nu\mathcal{A}^2\phi_{xx}$ changes in local speed (relative to fresh gas) of a flame front element [5]. The Hilbert transform is

$$\mathcal{H}\{\phi_x\}(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\phi_x(t, x')}{(x - x')} dx' \tag{2}$$

if written in a form suitable for the isolated crests ($\phi_x(t, \pm\infty) = 0$) to be first studied here; its form adapted to 2π-periodic patterns, to which (1) also applies, will be recalled in Section 5. This nonlocal term $\mathcal{H}\{\phi_x\}$ encodes the DL hydrodynamic flame instability [1–3]: since $\mathcal{H}(e^{i\kappa x}) = i.\mathrm{sgn}(\kappa)e^{i\kappa x}$ the growth/decay rate of normal modes $\phi \sim e^{i\kappa x + \varpi t}$ of the linearized (1) indeed reads $\varpi = |\kappa| - \nu\kappa^2$. The nonlinear contribution $-\frac{1}{2}\phi_x^2$ to ϕ_t mainly [6] is of geometric origin [1]: the normal to the front locally makes a finite angle $\gamma \sim -\mathcal{A}\phi_x$ to the mean propagation direction (normal to x-axis), and $\cos(\gamma) - 1 \sim -\mathcal{A}^2\phi_x^2/2$.

Although its dimensioned version was originally derived as a leading order result for $\mathcal{A} \to 0^+$ [1], the MS equation (1) still rules flame dynamics if corrections (removed from (1) by re-scaling) due to two more orders are retained in the $\mathcal{A}$-expansion [6,7] to improve accuracy. Besides gaseous combustion, the MS equation

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governs other unstable fronts coupled to Laplacian fields: in doped semi-conductors [8] or in reactive infiltration [9].

Eq. (1) exhibits a number of remarkable features, most notably the existence of pole-decompositions whereby the search for $\phi(t,x)$ is converted to a $2N$-body problem for the complex poles of the front slope [10,11]; see Section 2. Thanks to this property one can: (i) Explain the formation of front arches joined by sharper crests whose mergers ultimately produce the widest admissible steady cell; (ii) Access the latter's arc-length *vs.* wavelength curve [12], which yields the effective flame speed; (iii) Solve stability issues [13,14] without the effect of spurious noises hampering the non-self-adjoint linearized dynamics [15]; (iv) Compute pole density and front shapes for isolated crests, and then for periodic cells [11,16], if $N \gg 1$; (v) Study stretched crests [17]; (vi) Set up tools to study extensions of (1) that incorporate higher orders of the $\mathcal{A} \ll 1$ expansion, at least in the large-N limit ([18,19] and references therein).

It would be valuable to extend the latter works to *finite* pole numbers $2N$, as to encompass wrinkles of moderate wavelengths/amplitudes. New tools are needed and, just like in cases (iv)–(vi) above, studying solutions representing isolated front crests (localized bumps with $\phi_x(t,\pm\infty)=0, \phi(t,-x)=\phi(t,x)$) might be a key step to take up first. The present numerical exploratory approach finds that crest-type solutions of (1) have intriguing – though as yet unexplained – relationships with known polynomials; this will ultimately provide one with approximate crest shapes in closed form that are accurate even for finite Ns.

2. Pole-decomposition for isolated crests

As shown in [10,11] (1) admits solutions that have $\phi(t,x) = -2\nu\Sigma_{|k|=1,\ldots,N}\ln(x-z_k(t)) + \text{const.}$ whence:

$$\phi_x(t,x) = \sum_{k=-N}^{N} \frac{-2\nu}{x - z_k(t)}, \tag{3}$$

provided the complex-conjugate pairs of poles $z_k = \bar{z}_{-k}$, $|k| = 1,\ldots,N$, of the analytically-continued flame slope $\phi_x(t,z)$ in $z = x + iB$ plane satisfy:

$$\frac{dz_k}{dt} = \sum_{j=-N,j\neq k}^{+N} \frac{2\nu}{z_j - z_k} - i.\text{sgn}(\Im(z_k)), \tag{4}$$

where $\Im(.)$ is an imaginary part. The basic identity $(x-z_k)^{-1}(x-z_j)^{-1} = (x-z_k)^{-1}(z_k-z_j)^{-1} + (x-z_j)^{-1}(z_j-z_k)^{-1}$ indeed allows one to transform the cross-terms generated when squaring (3), and a similar one followed by contour integrations in z-plane leads to $\mathcal{H}[1/(x-z_k)] = -i.\text{sgn}(\Im(z_k))/(x-z_k)$. Combined with ϕ_t this transform (1) in a sum of $C_k/(x-z_k)$ pieces, and (4) are the conditions $C_k = 0$ for (1) to be satisfied. The solutions (3) are localized, $\phi_x(t,|x|/\nu \to \infty) \approx -4N\nu/x$, and represent a collection of elementary isolated flame front crests, each associated with by a pair z_k, z_{-k}. As implied by (3) $e^{-\phi(t,z)/2\nu}$ is a polynomial with the $z_n(t)$ as simple zeros; its degree, the total number $2N$ of poles, is conserved yet arbitrary.

It is a known consequence [11] of the $(z_k - z_j)^{-1}$ interaction terms in (4) that nearby poles undergo mutual vertical repulsion and horizontal attraction. This mechanism ultimately produces a single steady arrangement of vertically aligned poles, say located at $z_k = iB_k = -iB_{-k}$, $1 \le |k| \le N$. Such time-independent crest shapes $\phi(x)$ obey steady versions of (1)–(4):

$$\frac{1}{2}\phi_x^2 + \mathcal{H}\{\phi_x\} - \nu\phi_{xx} = 0, \tag{5}$$

$$\phi_x(x) = -\sum_{k=-N}^{N} \frac{2\nu}{x - iB_k}, \tag{6}$$

$$\sum_{j=-N,j\neq k}^{N} \frac{2\nu}{B_k - B_j} = \text{sgn}(B_k), \tag{7}$$

$$\phi(x) = -2\nu\ln(i^{2N}P_{2N}(x/i)) + \text{const.}, \tag{8}$$

where the even monic polynomial $P_{2N}(B) = B^{2N} + \cdots$ has the real pole 'altitudes' B_k as its zeros. The (dimensionless) Markstein [5] 'length' ν could obviously be scaled out from (7), but it is kept as is for future comparisons with other lengths besides the various B_k.

The $N = 1$ crest involves two poles $iB_{2,k}$, with $B_{2,\pm1} = \pm\nu$ obtained from (7) as zeros of:

$$P_2(B) \equiv B^2 - \nu^2. \tag{9}$$

For $N = 2$ the pole altitudes $B_{4,k} = \pm 3\nu(1 \pm 1/\sqrt{2})$ are the zeros of [11]:

$$P_4(B) \equiv B^4 - 27\nu^2B^2 + \frac{81}{4}\nu^4. \tag{10}$$

When $N = 3$ a tedious algebra is already required to write the polynomial with irrational coefficients $P_6(B)$ [11], not to mention formulae for its zeros. $P_{2N}(B)$ could not even be accessed analytically for $N > 3$; yet the zeros $B_{2N,k}$, $|k| = 1,\ldots,N$, can be obtained numerically for any N, by the Newton iterative resolution of (7) or as attractors of (4).

3. MS equation *vs.* MP polynomials

From the $B_{2N,k}$, $|k| = 1,\ldots,N$, computed on solving (7) for increasing Ns, monic polynomials $P_M(B) = B^M + \cdots$ are next defined by $P_0(B) = 1$, $P_1(B) = B$ and by (11) for $M = 2, 3, \ldots$:

$$P_{2N}(B) \equiv \prod_{k=1,..N} (B^2 - B_{2N,k}^2), \qquad P_{2N+1}(B) \equiv B \prod_{k=1,..N} (B^2 - b_{2N,k}^2). \tag{11}$$

The auxiliary b_k-zeros in there obey equations similar to (7), yet with an extra $b_0 = 0$ whose charge is -2ν, and for $|k| = 1,\ldots,N$ are roots of conditions of electrostatic-like equilibriums:

$$\frac{2\nu}{b_k - 0} + \sum_{j=-N,j\neq k}^{N} \frac{2\nu}{b_k - b_j} = \text{sgn}(b_k); \tag{12}$$

the equilibrium of $b_0 = 0$ is guaranteed by $b_{-k} = -b_k$. Such auxiliary $ib_{|k|\ge1} \neq 0$ are poles of a smooth slope profile $F_x(x)$ governed by $-\frac{2\nu}{x}F_x + \frac{1}{2}F_x^2 + \mathcal{H}(F_x) - \nu F_{xx} = 0$ instead of (5); with $b_0 = 0$ included, the ib_k are poles of a singular slope $f_x(x) := F_x(x) - \frac{2\nu}{x}$ obeying a locally-forced steady MS equation, *viz.* (5) with $2\pi\nu\delta(x)$ added to the right-hand side. The $b_{2N,k}$ were also computed numerically from (12), for increasing values of $N > 2$.

With the above convention on labels, two successive polynomials in the $\{P_M\}_{M=1,2\ldots}$ sequence have opposite parities, $P_M(-B) = (-1)^M P_M(B)$. From the numerically built sequence one next performs the Euclidean division of each $P_M(B)$ by its antecedent $P_{M-1}(B)$, to produce:

$$P_M(B) = BP_{M-1}(B) - C(M)R_{M-2}(B), \tag{13}$$

where each polynomial remainder $R_{M-2}(B)$ has degree $M-2$, parity $(-1)^{M-2}$, and may also be assumed monic on defining the coefficient $C(M)$ accordingly.

The numerically computed zeros $\beta_{M-2,k}$ of $R_{M-2}(B)$ happened to be real for all M, which was not obvious, and, surprisingly enough, the roots $\beta_{2N-2,k}$ of $R_{2N-2}(B) = 0$ were found to nearly coincide with the zeros, $B_{2N-2,k}$, of the 'Sivashinsky polynomials' $P_{2N-2}(B)$: the fractional differences are less than 0.2% for N as low as 11, see Table 1, and in all cases look smaller than typical $\mathcal{O}(1/N)$ quantities. Likewise, the auxiliary poles $b_{2N-1,k}$ were found to sit very close to the roots of $R_{2N-1}(B) = 0$.