Physica D 292–293 (2015) 70–83

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Mixed mode oscillations in a conceptual climate model

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HIGHLIGHTS

- A conceptual model for Pleistocene glacial cycles is developed.
- The model is analyzed using geometric methods for multiple time-scale systems.
- For certain parameters the model exhibits mixed-mode oscillations.

ARTICLE INFO

Article history:
Received 25 February 2014
Received in revised form
28 October 2014
Accepted 7 November 2014
Available online 15 November 2014
Communicated by H.A. Dijkstra

Keywords:
Mixed-mode oscillations
MMO
Fast/slow
Folded node
Glacial cycle
Paleoclimate

ABSTRACT

Much work has been done on relaxation oscillations and other simple oscillators in conceptual climate models. However, the oscillatory patterns in climate data are often more complicated than what can be described by such mechanisms. This paper examines complex oscillatory behavior in climate data through the lens of mixed-mode oscillations. As a case study, a conceptual climate model with governing equations for global mean temperature, atmospheric carbon, and oceanic carbon is analyzed. The nondimensionalized model is a fast/slow system with one fast variable (corresponding to ice volume) and two slow variables (corresponding to the two carbon stores). Geometric singular perturbation theory is used to demonstrate the existence of a folded node singularity. A parameter regime is found in which (singular) trajectories that pass through the folded node are returned to the singular funnel in the limiting case where $\epsilon = 0$. In this parameter regime, the model has a stable periodic orbit of type 1^s for some $s > 0$. To our knowledge, it is the first conceptual climate model demonstrated to have the capability to produce an MMO pattern.

1. Introduction

There has been a significant amount of research aimed at explaining oscillations in various historical periods of the climate system. Saltzman and Maasch have a series of papers on the Mid-Pleistocene transition, a change from oscillations with a dominant period of 40 kyr to oscillations with a dominant period of 100 kyr [1–3]. According to Saltzman and Maasch, the 40 kyr oscillations in the data result from a linear response to (quasi-)periodic changes in astronomical forcing. They propose that the transition to the 100 kyr cycles occurs due to a Hopf bifurcation producing an attracting periodic orbit. Paillard and Parrenin also seek to explain the Mid-Pleistocene transition and the glacial–interglacial cycles of the late Pleistocene, with a discontinuous and piecewise linear model [4]. Their work, and the work of Hogg [5], use changes in astronomical forcing due to variation in the Earth's orbit to generate oscillations. Crucifix [6] and Ditlevsen [7] review oscillations in conceptual climate models. In particular, Crucifix [6] discusses relaxation oscillators in ice-age models. However, to our knowledge, the discussion is limited to single-amplitude or single mode oscillations.

Looking at Fig. 1, each 100 kyr cycle contains a sharp increase leading into the interglacial period (denoted by the red spikes). This relaxation behavior clearly indicates the existence of multiple time-scales in the underlying problem. There are also smaller, structured oscillations in the glacial state that are repeated in each 100 kyr cycle. The presence of the large relaxation oscillation and the small amplitude oscillations indicates that these may be

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http://dx.doi.org/10.1016/j.physd.2014.11.003
0167-2789/

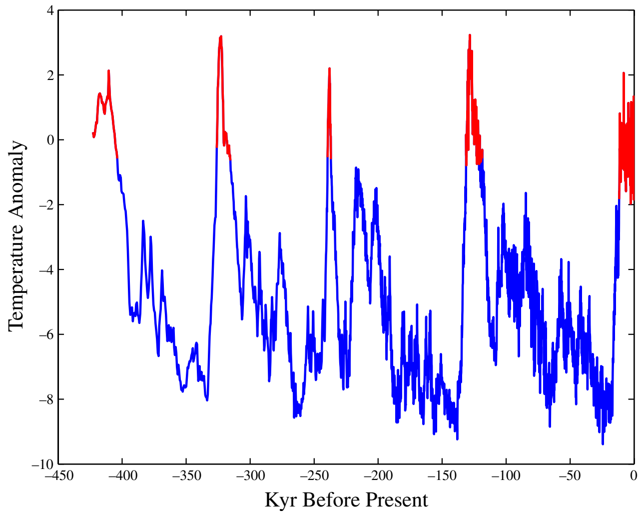

Fig. 1. Temperature anomaly obtained from the Vostok ice core deuterium record [9]. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

mixed-mode oscillations (MMOs)—a pattern of L_1 large amplitude oscillations (LAOs) followed by s_1 small amplitude oscillations (SAOs), then L_2 large spikes, s_2 small cycles, and so on. The sequence ${L_1}^{s_1}{L_2}^{s_2}{L_3}^{s_3}\dots$ is known as the MMO signature [8]. We propose that the 100 kyr glacial–interglacial cycles as well as the largest of the SAOs – i.e., the largest cycles that do not enter the interglacial state – can be interpreted as MMOs. By suggesting that the SAOs result from intrinsic dynamics, we are proposing an alternative to the standard interpretation that attributes them to changes in astronomical forcing. We note, however, that neither interpretation necessarily rules out the other—it may be possible to combine both intrinsic and forced oscillations.

This paper tests the scientific hypothesis that oscillatory behavior in climate data can be interpreted as MMOs, and we take the data in Fig. 1 as a case study. Desroches et al. survey the mechanisms that can produce MMOs in systems with multiple time-scales [8]. From the data set shown in Fig. 1, we know that the underlying model has a multiple time-scale structure. If we want to find MMOs, the model must have at least three state variables. Assuming we can find a global time-scale splitting, there are three distinct ways to have a 3D model with multiple time scales: (a) 1 fast, 2 slow; (b) 2 fast, 1 slow; and (c) 1 fast, 1 intermediate, 1 slow (i.e., a three time-scale model). Each of these options can create MMOs through different mechanisms. Models with 1 fast and 2 slow variables can create MMOs through a folded node or folded saddle–node with a global return mechanism that repeatedly sends trajectories near the singularities. Models with 1 slow and 2 fast variables can create MMOs through a delayed Hopf mechanism that also requires a global return. MMOs in three time-scale models are reminiscent of MMOs due to a folded saddle–node type II – where one of the equilibria is a folded singularity – although the amplitudes of the SAOs are more pronounced in this case.

To verify our hypothesis, we have to strike a delicate balance. The model needs to be complex enough to exhibit the desired behavior, but if it is too complex we will be unable to *prove* that it does so. We know from the data shown in Fig. 1 that temperature shows relaxation behavior, rapidly oscillating between two meta-stable states. Assuming that ice volume is strongly correlated with temperature, we view glacial cycles in the same way, i.e., oscillating between two meta-stable states. The possibility of a bistable regime in ice volume has been discussed for decades, notably by Weertman [10], MacAyeal [11], Oerlemans [12], Calov and Ganopolski [13], Crucifix [14], and Abe-Ouchi et al. [15]. Atmospheric carbon should also play a role in any model that describes glacial–interglacial cycles, as suggested by Saltzman and Maasch [3] as well as Paillard and Parrenin [4]. We consider a physical, conceptual model that incorporates continental ice sheets, atmospheric carbon, and oceanic carbon. Since this approach has never been used in a climate-based model, our desire is that the analysis is clear enough to replicate. This is a major reason for our choice of such a simplistic 3D model. Indeed, we omit time-dependent forcing such as Milankovitch cycles, leaving these effects to future work. Even so, a minimal model is able to provide insight into key mechanisms behind the MMOs. We include oceanic carbon as the third variable because the model was able to produce MMOs. However, we were unable to find MMOs in other minimal models with, for example, deep ocean temperature.

Our analysis will rely heavily on the model and ideas put forth by MacAyeal in [11] where the physical units – as well as the physical meaning of some parameters – are ambiguous. His approach to explaining glacial cycles with a catastrophe model is similar to our MMO approach, without the benefit of 25 years of mathematical development. Rather than using independently varying parameters as a means of generating slow dynamics, we couple MacAyeal's model with (simplified) carbon dynamics. The main task is to obtain the "global" time-scale separation between the ice sheet evolution and the evolution of the carbon equations denoted by ϵ_1 and ϵ_2. In general, a time-scale separation can be revealed through dimensional analysis. The process should relate a small parameter ϵ_i to physical parameters of the dimensional model. In applications such as neuroscience, it is often possible to get a handle on the "smallness" of the ϵ_i because there are accepted values or ranges for many of the physical parameters. Unfortunately, parameters in paleoclimate models are not as constrained. We rely on the intuition of physicists, geologists, and atmospheric scientists to determine a reasonable separation of time-scales.

While it may be unsettling to not have a more concrete argument, the ambiguity regarding parameter values – and even the governing equations – allows more freedom. With this in mind we take a different approach than that of others in the paleoclimate literature such as Saltzman and Maasch [1–3]. In the vast majority of climate science papers, the authors simulate models with judiciously chosen parameters. Our approach is different in that we assume nothing about any parameters except that they are physically meaningful. Then, through the analysis, we find conditions under which the model behaves qualitatively like the data. The idea is not to pinpoint specific parameter values, but to find a range of possible parameters. There are two advantages to this approach. First, the parameter range can be used to constrain (or maybe constrain further) previous parameter estimates, which may tell us something previously unknown about the climate system. It can be used to inform parameter choices for large simulations. Second, a parameter range is useful to eliminate options. That is, if the only parameter range which produces the correct qualitative behavior is entirely unreasonable, the model needs to be changed.

The outline of the paper is as follows: In Section 2 we set up the model and provide relevant background from the paleoclimate literature. Then we nondimensionalize the model and discuss assumptions on some of the parameters. In particular, we identify our dimensionless model as a multiple time-scale problem. We analyze this model in Section 3, with a focus on finding conditions for MMOs. We conclude with a discussion in Section 4.

2. Setting up the model

2.1. The physical model

We start with a model of the form

$$\gamma \frac{dX_e}{dt} = A_0(B_0 - A) - B_1 X_e^3 + B_2 X_e \tag{1}$$