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The effect of boundaries on the asymptotic wavenumber of spiral wave solutions of the complex Ginzburg–Landau equation

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HIGHLIGHTS

- n-armed spiral waves of the complex Ginzburg–Landau equation in a disk are considered.
- We study the effect of boundaries on the asymptotic wavenumber k.
- We obtain formulas for k as a function of the domain radius and the twist parameter.
- Small radius have strong effects on the asymptotic wavenumber.
- Large radius have exponentially small effects on the asymptotic wavenumber.

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ABSTRACT

In this paper we consider an oscillatory medium whose dynamics are modeled by the complex Ginzburg–Landau equation. In particular, we focus on n-armed spiral wave solutions of the complex Ginzburg–Landau equation in a disk of radius d with homogeneous Neumann boundary conditions. It is well-known that such solutions exist for small enough values of the twist parameter q and large enough values of d. We investigate the effect of boundaries on the rotational frequency of the spirals, which is an unknown of the problem uniquely determined by the parameters d and q. We show that there is a threshold in the parameter space where the effect of the boundary on the rotational frequency switches from being algebraic to exponentially weak. We use the method of matched asymptotic expansions to obtain explicit expressions for the asymptotic wavenumber as a function of the twist parameter and the domain size for small values of q.

1. Introduction

Rotating spiral waves are commonly found in many physical, chemical and biological systems both in excitable and oscillatory media (c.f. [1–9]). Excitable media are characterized by producing oscillations only in the presence of waves, that is to say, there has to be some steady perturbation or some inhomogeneity in the medium in order for spiral waves to exist. Such systems are usually described in terms of a Fitzhugh–Nagumo system (c.f. [10–14]), that is a system of partial differential equations with a unique stable fixed point. However, oscillatory systems are able to produce oscillations in homogeneous situations. This is in general the case of reaction–diffusion systems with a scalar diffusion that evolve in the vicinity of a Hopf bifurcation. By formally expanding such systems near the critical parameter, the celebrated complex Ginzburg–Landau equation is obtained (see for instance Section 2 in [15]),

$$\frac{\partial \Psi}{\partial t} = \Psi - (1+ia)|\Psi|^2\Psi + (1+ib)\nabla^2\Psi, \tag{1}$$

where a and b are real parameters and Ψ is a complex field representing the amplitude and phase of the modulations of the oscillatory pattern. The class of solutions that we study in this paper are rotational solutions of (1) with a given frequency ω. Factoring out this rotation and introducing the following scalings in (1)

$$\Psi = e^{-i\omega t}\sqrt{\frac{1+\omega b}{1+ab}}\,\psi, \qquad t = \frac{t'}{1+\omega b},$$

$$(x,y) = \sqrt{\frac{1+b^2}{1+b\omega}}(x',y'),$$

gives

$$(1-ib)\frac{\partial \psi}{\partial t'} = (1-|\psi|^2)\psi + iq\psi(1-k^2-|\psi|^2) + \nabla^2\psi, \tag{2}$$

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where $q = (a-b)/(1+ab)$ and k is such that $q(1-k^2) = (\omega - b)(1+b\omega)$. The parameter q is usually known as the *twist parameter*. When $q = 0$, spiral wave solutions of (2) have isophase lines that are simple straight lines emanating from the origin (see [16,17] for more details). The parameter k is known as the *asymptotic wavenumber*, as it represents a wavenumber at infinity as we will show later. In this sense the expression $q(1-k^2) = (\omega-b)(1+b\omega)$ provides a dispersion relation for the spiral wave.

In this paper we focus on single n-armed spiral waves of (1) in a disk of radius d finite, which are solutions of (2) of the form

$$\psi = f(r)e^{i(n\phi+\varphi(r))}, \tag{3}$$

where r and ϕ are the polar radius and azimuthal coordinates of the plane and $n \in \mathbb{Z}$ is the degree or winding number. This type of solution corresponds to a single spiral emanating from the center of the domain. It is required that $f(0) = 0$ so that it is continuous at the origin.

The same type of solutions in unbounded domains have been thoroughly studied by several authors. For example Kopell and Howard [18], rigorously prove that, for each value of the twist parameter q, there exist solutions of the form (3) only for a particular value of the frequency ω and therefore for a particular asymptotic wavenumber k. Hagan in [16] studies the problem for small q in the whole plane using the method of matched asymptotics and finds that the asymptotic wavenumber k is exponentially small in q. In all these works the phase function $\varphi(r)$ is found to be $\varphi(r) \sim -kr$ as $r \to \infty$, so the isophase lines (far enough from the center of the spiral) are of the form of Archimedean spirals, that is $n\phi - kr$ is equal to constant. This is the reason why the constant k is usually known as the asymptotic wavenumber. However, in experiments or numerical simulations the domain is bound to be finite, and the key question is whether the boundaries do alter the spiral shape or even prevent its existence. The natural boundary conditions to impose in both experiments and numerical computations are zero flux boundary conditions, that is $\partial_\nu \Psi|_{\partial\Omega} = 0$. Written in terms of $f(r)$ and $\varphi(r)$, this becomes $f'(d) = 0$ and $\varphi'(d) = 0$, where d is the radius of the finite disk. In this case, one can write a set of ordinary differential equations for $f(r)$ and $\varphi(r)$ by substituting (3) in (2)

$$0 = f'' + \frac{f'}{r} - (\varphi')^2 f - f\frac{n^2}{r^2} + f(1-f^2), \tag{4}$$

$$0 = f\varphi'' + 2\varphi' f' + \frac{\varphi' f}{r} + qf(1-k^2-f^2). \tag{5}$$

Due to the gauge invariance of solutions of (1), the phase function φ does only appear in these equations through its derivative, so that (4)–(5) is effectively a third order system of ordinary differential equations for $f(r)$, $f'(r)$ and $\varphi'(r)$. It is therefore convenient to write $v(r) = \varphi'(r)$ which yields the system

$$0 = f'' + \frac{f'}{r} - v^2 f - f\frac{n^2}{r^2} + f(1-f^2), \tag{6}$$

$$0 = fv' + 2vf' + \frac{vf}{r} + qf(1-k^2-f^2), \tag{7}$$

with the following set of boundary conditions,

$$f(0) = v(0) = 0, \qquad f'(d) = v(d) = 0. \tag{8}$$

Paullet et al. [17] show that the system (6)–(7) along with the boundary conditions (8) has a unique solution only for small enough values of q and for radius d of order greater than one. Furthermore, they show that such solution does only exist for a particular value of k. They use a two parameter shooting argument, with the rotational frequency ω and the derivative of f at the origin as the shooting parameters. Furthermore, based on their numerical results, they conjecture that for 1-armed spirals, the critical size of the domain is given by $d = j_1 = 1.841183\ldots$, where j_n is the first positive zero of the first derivative of the Bessel function of the first kind of order one. Later, Tsai in [19] provides a rigorous proof (for a larger class of systems which include the complex Ginzburg–Landau one) which states that, in general, n-armed spiral waves do not exist if $d \le j_n$. Furthermore, they show that if $d > j_n$, there exists a frequency ω or, equivalently, a value of the asymptotic wavenumber k, for which there exists a spiral wave solution of (2).

However, none of these papers address the problem of how the domain size changes the rotational frequency and thus the asymptotic wavenumber of the spirals. Numerical simulations presented in [19], suggest that, in the limit of small twist parameter, the rotational frequency depends strongly on the domain size, while this dependency weakens as the domain becomes larger. In particular, the rotational frequency in large domains approaches that of infinite domains obtained by Hagan in [16]. The main contribution of this paper is precisely to provide a set of formulas for the asymptotic wavenumber as a function of the twist parameter q and the size of the domain d, for small values of q. In particular, we show that the curve $q|n|\log d = \pi/2$ separates the parameter space into two regions where the spirals and their corresponding asymptotic wavenumber change substantially:

- If $q|n|\log d \gtrsim \pi/2$, we find that

$$k = \frac{2}{q}\exp\left(-\gamma + c_n - \frac{\pi}{2q|n|} + \frac{K_1(kqd)}{I_1(kqd)}\right)(1+o(1)), \tag{9}$$

  where c_n depends on the winding number n but is independent of q and d. Here K_1 and I_1 are the modified Bessel functions of the first and the second kind of order one. This expression shows that the effect of the domain size on k and therefore, on the rotational frequency, is indeed exponentially weak. In particular we further show that this equation, in the limit as $d \to \infty$ yields

$$k = \frac{2}{q}\exp\left(-\gamma + c_n - \frac{\pi}{2q|n|}\right)(1+o(1)), \tag{10}$$

  which corresponds to the asymptotic wavenumber of a single spiral wave in an infinite domain first obtained by Hagan in [16].
- If $q|n|\log d \ll \pi/2$, the asymptotic wavenumber has a stronger dependence on the domain size given by the following expression:

$$k = \frac{1}{qd}\left(2q|n|\tan(q|n|\log d)\right)^{1/2}(1+o(1)).$$

To obtain these expressions we use the method of matched asymptotic expansions. We use a two-parameter expansion in the two small parameters of the problem, namely q and $\epsilon = 1/d$. Following the standard procedure, we first construct a solution close to the spiral's center or spiral's core (the so-called *inner* solution), which we then match to the solution outside of the spiral core. The latter is known as the *outer* solution. However, dealing with two parameters rather than one causes a set of difficulties in the matching procedure which should be addressed differently depending on whether $q|n|\log d$ is close or far from the already mentioned threshold value $\pi/2$.

The paper is organized as follows. First, in Section 2 we consider a simpler one-parameter expansion in q while keeping $\epsilon = 1/d$ as an order one quantity. We find that the resulting asymptotic formula for k breaks down as soon as the domain size d becomes greater than one. However, according to [17,19], the size of the domain must always be greater than $j_n > 1$, so the actual domains where spiral wave solutions exist are effectively of order greater than one, which shows that the solutions in Section 2 are indeed useless in practice. Hence, in Section 3 we consider the full two-parameter expansion when both q and ϵ are small and obtain the expressions for the asymptotic wavenumber (9) and (10). We start