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Shearless transport barriers in unsteady two-dimensional flows and maps

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HIGHLIGHTS

- A variational theory is developed for shearless transport barriers in unsteady flows.
- Shearless barriers are shown to be special null-geodesics of a Lorentzian metric.
- We devise an algorithm for automated detection of shearless barriers.
- The algorithm is tested on the standard non-twist map.
- Shearless barriers of a chaotically forced Bickley jet are studied.

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ABSTRACT

We develop a variational principle that extends the notion of a shearless transport barrier from steady to general unsteady two-dimensional flows and maps defined over a finite time interval. This principle reveals that hyperbolic Lagrangian Coherent Structures (LCSs) and parabolic LCSs (or jet cores) are the two main types of shearless barriers in unsteady flows. Based on the boundary conditions they satisfy, parabolic barriers are found to be more observable and robust than hyperbolic barriers, confirming widespread numerical observations. Both types of barriers are special null-geodesics of an appropriate Lorentzian metric derived from the Cauchy–Green strain tensor. Using this fact, we devise an algorithm for the automated computation of parabolic barriers. We illustrate our detection method on steady and unsteady non-twist maps and on the aperiodically forced Bickley jet.

1. Introduction

Consider a two-dimensional dynamical system with a family of invariant closed curves that are formed by periodic or quasi-periodic trajectories. The trajectories trace the invariant curves at specific frequencies. A shearless transport barrier then is generally defined as the invariant curve whose frequency admits a local extremum within the family. This definition ties shearless barriers fundamentally to recurrent (i.e., steady, periodic or quasi-periodic) flows where the necessary frequencies are well-defined. Here we extend the notion of a shearless transport barrier to two-dimensional flows and maps with general time-dependence.

In steady and time-periodic problems of fluid dynamics and plasma physics, shearless (or non-twist) barriers have been found to be particularly robust inhibitors of phase space transport [1–4]. For illustration, consider a steady, parallel shear flow

$$\dot{x} = u(y), \qquad u'(y_0) = 0. \qquad (1)$$
$$\dot{y} = 0,$$

on a domain periodic in x. The $y = y_0$ line marks a jet core, whose impact on tracer patterns is shown in Fig. 1 in a particular example with $y_0 = 0$. Note the unique material signature of the shearless barrier, deforming the tracer blob initialized along it into a boomerang-shaped pattern, by contrast, another tracer blob simply stretches under shear.

The flow (1) is an idealized model of the velocity field inside atmospheric or oceanic zonal jets, or helical magnetic field lines in a tokamak [5]. As a dynamical system, (1) represents an integrable

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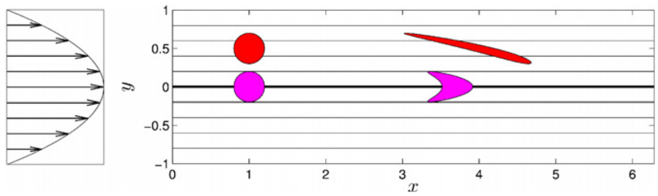

Fig. 1. Left: The velocity profile of the steady flow (1) for $u(y) = 1 - y^2$. Right: Streamlines for the same flow. The thick line at $y = 0$ marks the shearless streamline that acts as a jet core. The tracer disk located on the shearless line (magenta circle) deforms into a blunt arrow shape symmetrically under advection to time $t = 9$. The tracer disk located away from the shearless line (red circle) has a markedly different deformation pattern. The boundary condition in the x-direction is taken to be periodic with period 2π. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

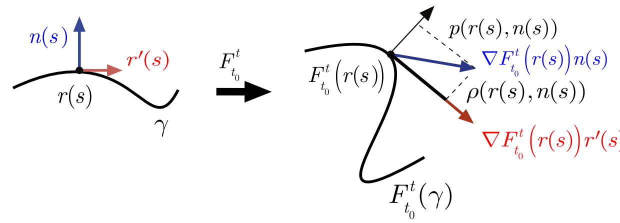

Fig. 2. The evolution of a unit normal vector $n(s)$ of a material line γ under the linearized flow map $\nabla F_{t_0}^t$.

system with the Hamiltonian $H(y) = \int_0^y u(\eta)\mathrm{d}\eta$. Its horizontal trajectories along which the Eulerian shear $u'(y)$ vanishes are referred to as shearless barriers. Along these barriers, $H''(y_0) = 0$ holds, thus the circle $y = y_0$ does not satisfy the twist condition of classic KAM theory [6].

Yet numerical studies of [1–3,7] show that such barriers are more robust under steady or time-periodic perturbations than any other nearby KAM tori. Related theoretical results for two-dimensional maps were given in [8]. More recently, degenerate tori for steady 3D maps were considered in [9]. In addition, a general *a posteriori* result on non-twist tori of arbitrary dimension that are potentially far from integrable has been obtained by [10]. However, no general theory of shearless transport barriers for unsteady flows has been established.

The need for such a general theory of unsteady shearless barriers clearly exists. In plasma physics, computational and experimental studies suggest that shearless barriers enhance the confinement of plasma in magnetic fusion devices [11–14], which generate turbulent velocity fields with general time dependence. In this context, a description of shearless barriers is either understood in models for steady magnetic fields [14] or inferred from scalar quantities (e.g. temperature, density) in more complex unsteady scenarios [11–13].

In fluid dynamics, shearless barriers are of interest in the context of zonal jets. Rossby waves are the best known and most robust transport barriers in geophysical flows [15–17], yet only recent work attempts to describe their attendant unsteady jet cores in the Lagrangian frame of an unsteady flow. The method put forward in [18] seeks such Lagrangian shearless barriers as trenches of the finite-time Lyapunov exponent (FTLE) field. However, just as the examples in [19] show that FTLE ridges do not necessarily correspond to hyperbolic Lagrangian structures, FTLE trenches may also fail to mark zonal jet cores (see Example 1 in Section 7.2 below).

Here we develop a variational principle for shearless barriers as centerpieces of material strips showing no leading order variation in Lagrangian shear. This variational principle shows that shearless barriers are composed of tensorlines of the right Cauchy–Green strain tensor associated with the flow map. Most stretching or contracting Cauchy–Green tensorlines have previously been identified as best candidates for hyperbolic Lagrangian Coherent Structures (LCSs) [20,21], but no underlying global variational principle has been known to which they would be solutions. The present work, therefore, also advances the theory of hyperbolic LCS, establishing them as shearless transport barriers under fixed (Dirichlet-type) boundary conditions.

Our main result is that parabolic transport barriers (jet cores) are also solutions of the same shearless Lagrangian variational principle, satisfying variable-endpoint boundary conditions. They are formed by minimally hyperbolic, structurally stable chains of tensorlines that connect singularities of the Cauchy–Green strain tensor field. We develop and test a numerical procedure that detects such tensorline chains, thereby finding generalized Lagrangian jet cores in an arbitrary, two-dimensional unsteady flow field in an automated fashion.

2. Notation and definitions

Let $v(x, t)$ denote a two-dimensional velocity field, with x labeling positions in a two-dimensional region U, and with t referring to time. Fluid trajectories generated by this velocity field satisfy the differential equation

$$\dot{x} = v(x, t), \tag{2}$$

whose solutions are denoted by $x(t; t_0, x_0)$, with x_0 referring to the initial position at time t_0. The evolution of fluid elements is described by the flow map

$$F_{t_0}^t(x_0) := x(t; t_0, x_0), \tag{3}$$

which takes any initial position x_0 to its current position at time t.

Lagrangian strain in the flow is often characterized by the right Cauchy–Green strain tensor field $C(x_0) = \left[\nabla F_{t_0}^t(x_0)\right]^T \nabla F_{t_0}^t(x_0)$, whose eigenvalues $\lambda_i(x_0)$ and eigenvectors $\xi_i(x_0)$ satisfy

$$C\xi_i = \lambda_i \xi_i, \quad |\xi_i| = 1, \ i = 1, 2; \qquad 0 < \lambda_1 \le \lambda_2, \qquad \xi_1 \perp \xi_2.$$

The tensor C, as well as its eigenvalues and eigenvectors, depend on the choice of the times t and t_0, but we suppress this dependence for notational simplicity.

3. Stability of material lines

Consider a material line (i.e., a smooth curve of initial conditions) γ at time t_0, parametrized as $r(s)$ with $s \in [0, \sigma]$. If $n(s)$ denotes a smoothly varying unit normal vector field along γ, then the *normal repulsion* ρ of γ over the time interval $[t_0, t]$ is given by [19]

$$\rho(r, n) = \frac{1}{\sqrt{\langle n, C^{-1}(r)n\rangle}}, \tag{4}$$

measuring at time t the normal component of the linearly advected normal vector $\nabla F_{t_0}^t(r)n$ (see Fig. 2). If $\rho > 1$ pointwise along γ, then the evolving material line $F_{t_0}^t(\gamma)$ is repelling. Similarly, if $\rho < 1$ holds pointwise along γ, then the evolving material line $F_{t_0}^t(\gamma)$ is attracting.

Hyperbolic Lagrangian coherent structures (LCSs) are pointwise most repelling or most attracting material lines with respect to small perturbations to their tangent spaces [19,22,21]. Repelling and attracting LCSs, respectively, are obtained as special trajectories of the differential equations

$$\dot{r} = \xi_1(r), \qquad \dot{r} = \xi_2(r), \tag{5}$$

that stay bounded away from points where ξ_i cease to be well-defined. These degenerate points x_0 are singularities of the Cauchy–Green tensor field, satisfying $C(x_0) = \lambda I$ for some $\lambda > 0$. (For an incompressible flow we have $\lambda = 1$.) The trajectories of the