Physica D 275 (2014) 1–7

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Functional relation between fluctuation and node degree in coupled stochastic dynamical systems

Woo-Sik Son, Dong-Uk Hwang, Jong-Ho Kim *

National Institute for Mathematical Sciences, Daejeon 305-390, Republic of Korea

HIGHLIGHTS

- Novel approaches using the average of higher order moments for the functional relation between noisy fluctuation and node degree.
- Improving the accuracy of functional relation for the network with strong heterogeneity in degree distribution.
- Investigation of the functional relation between noisy fluctuation and node input strength.

ARTICLE INFO

Article history:
Received 28 October 2013
Received in revised form
4 February 2014
Accepted 4 February 2014
Available online 12 February 2014
Communicated by K. Josic

Keywords:
Fluctuation versus node degree
Coupled stochastic dynamical system
Inverse problem of network property

ABSTRACT

For the coupled stochastic dynamical system, we study the functional relation between noisy fluctuation and node degree. We extend the approaches for obtaining functional relation in Wang et al. (2009) to the weighted network whose link weight is dependent on the node degree. For the network with strong heterogeneity in degree distribution, we find that the theoretical result derived from the approaches in Wang et al. (2009) shows disagreement with numerical results. Here, we propose novel approaches using the average of higher order moments and improve the accuracy of functional relation between noisy fluctuation and node degree. Also, we investigate the functional relation of noisy fluctuation versus node input strength.

1. Introduction

Recently there have been growing interests for inferring the network structure from measuring nodal time series. The source of these interests arises from many scientific areas. Examples are interacting proteins or genes [1], complex brain networks [2], ecological food webs [3], to name a few. In this regard, Yu et al. [4] showed that the topology of network can be estimated using the autosynchronization method. The autosynchronization method is introduced by Parlitz [5] and the parameters of controlled system converge to those of real system via appropriate control signal. Later, an extension of Ref. [4] was proposed based on a general LaSalle's principle [6]. By applying external driving and measuring the response dynamics, Timme [7] reconstructed the connectivity of coupled oscillator network. Also, the L1 norm optimization applied to estimate the topology of networks whose connections are sparse [8]. In the presence of noise, Ren et al. [9] showed that the network connectivity can be successfully reconstructed by the dynamical correlation of nodes.

With approximated numerical derivatives based on multivariate time series, the problem of inferring network structure can be casted into the problem of finding the solution of simultaneous equations. From this point of view, the topology of network was reconstructed by the error function minimization for the overdetermined system [10] and by compressive sensing for the underdetermined system [11]. Through repeatedly reinitializing the dynamics by random phase resetting, the network structure was obtained by averaging over the ensemble [12]. Moreover, Kim et al. [13] showed that the network link weights can be estimated from the inverse phase synchronization indices.

Concerned with the work of Ren et al. [9], it was shown that noisy fluctuation $\langle \Delta x_j^2 \rangle_T$ scales with the node degree k_j as $\langle \Delta x_j^2 \rangle_T \sim k_j^{-1}$ [14]. Here the noisy fluctuation is defined as $\langle \Delta x_j^2 \rangle_T = \langle (x_j - \langle x \rangle_E)^2 \rangle_T$, where $\langle \cdot \rangle_E$ and $\langle \cdot \rangle_T$ denote the ensemble and time average, respectively. This scaling relation indicates that for higher degree node such as hub, the node shows smaller fluctuation from the ensemble average. But, for smaller degree node such as periphery, the node shows larger fluctuation. It holds for a variety of network topologies and node dynamics. Using the re-

* Corresponding author. Tel.: +82 42 717 5742; fax: +82 42 717 5734.
E-mail addresses: wsson@nims.re.kr (W.-S. Son), duhwang@nims.re.kr (D.-U. Hwang), jhkim@nims.re.kr (J.-H. Kim).

http://dx.doi.org/10.1016/j.physd.2014.02.002
0167-2789/

lation between noisy fluctuation and node degree, the authors of Ref. [14] can estimate the degree of node from nodal time series.

On the other hand, it was known that strong heterogeneity in degree distribution could reduce the synchronizability [15]. To overcome this heterogeneity, Motter et al. [16] proposed to scale the coupling strength by the power of node degree. That is, the element of coupling matrix is represented by $G_{jl} = L_{jl}/k_j^{\beta}$. Here, L is Laplacian matrix of underlying network: the off-diagonal entry is defined to be $L_{jl} = -1$ if there exists a link from node l to j, $L_{jl} = 0$ otherwise. The diagonal element is represented by $L_{jj} = k_j = -\sum_{l=1,l\neq j}^{N} L_{jl}$ and β is a tunable parameter. Although Motter et al. [16] introduced the coupling scheme $G_{jl} = L_{jl}/k_j^{\beta}$ to decrease the heterogeneity of degree, the role of a hub and peripheral node can be exchanged by altering the value of $\beta(0, 1, 2)$. When $\beta = 0$, G returns to intrinsic underlying network L. For the case of $\beta = 1$, all nodes get same input strength $s_{in}^{j} = -\sum_{l=1,l\neq j}^{N} G_{jl} = -\sum_{l=1,l\neq j}^{N} L_{jl}/k_j = 1$. Otherwise, when $\beta = 2$, the node input strength is given by $s_{in}^{j} = -\sum_{l=1,l\neq j}^{N} G_{jl} = -\sum_{l=1,l\neq j}^{N} L_{jl}/k_j^2 = 1/k_j$. Then, for the case of $\beta = 2$, higher degree node gets smaller influence from network. While lower degree node gets larger influence. The case of $\beta = 2$ might be considered as unusual coupling scheme. However, the hub does not always drive the periphery. On the contrary to ordinary case of $\beta = 0$, the information can flow from the periphery to hub, not from the hub to periphery [17]. Note that the distribution of node input strength becomes more heterogeneous in the case of $\beta = 2$. Because nodes either have a small number of very strong connections or a large number of very weak connections. However, our motivation is not to decrease the heterogeneity of input strength but to investigate the change of functional relations of noisy fluctuation versus node degree and versus node input strength by changing the role of a hub and peripheral node.

In this paper, we find that the functional relation derived from Ref. [14] shows disagreement with numerical results for the network with strong heterogeneity in degree distribution. Then, we propose novel theoretical approaches for obtaining the functional relation and show that it is more accurate than the results derived from the approaches in Ref. [14]. The remaining of this paper is organized as follows. In Section 2, we describe the model of coupled stochastic dynamical system. Section 3 presents the theoretical results derived from the approaches in Ref. [14]. In Section 4, we propose novel theoretical approaches and improve the accuracy of functional relation between noisy fluctuation and node degree. Conclusions are given in Section 5.

2. Model

We consider the coupled stochastic dynamical system described as

$$\dot{\mathbf{x}}_j = \mathbf{F}(\mathbf{x}_j) - c\sum_{l=1}^{N} G_{jl}\mathbf{H}(\mathbf{x}_l) + \xi_j, \tag{1}$$

where $\mathbf{x}_j$ is a multi-dimensional variable of j_{th} node and $\mathbf{F}$ denotes the node dynamics. c, $\mathbf{H}$, and G_{jl} are a coupling strength, coupling function, and element of coupling matrix (L_{jl}/k_j^{β}), respectively. In the remaining portion of the paper, we only consider the symmetric underlying network L. The Gaussian white noise ξ_j with intensity σ^2 satisfies

$$\langle \xi_j(t)\rangle_T = 0, \qquad \langle \xi_j(t)\xi_l(t')\rangle_T = \sigma^2\delta_{jl}\delta(t-t'). \tag{2}$$

Here, we deal with four different types of networks: Erdős–Rényi (ER) [18], Barabási–Albert (BA) [19], Watts–Strogatz (WS) [20], and configuration model (CM) [21,22].

As the node dynamics, the following three dynamical processes are considered: consensus dynamics [23], Rössler system [24], and Kuramoto oscillators [25]. In the presence of noise, described by the adjacency matrix representation ($A_{jl} = -L_{jl}, A_{jj} = 0$), the consensus dynamics is written by

$$\dot{x}_j = c\sum_{l=1}^{N} A_{jl}k_j^{-\beta}(x_l - x_j) + \xi_j. \tag{3}$$

For the chaotic dynamics, we consider Rössler system described as

$$\begin{aligned}
\dot{x}_j &= -y_j - z_j + c\sum_{l=1}^{N} A_{jl}k_j^{-\beta}(x_l - x_j) + \xi_j,\\
\dot{y}_j &= x_j + 0.2\,y_j + c\sum_{l=1}^{N} A_{jl}k_j^{-\beta}(y_l - y_j),\\
\dot{z}_j &= 0.2 + z_j(x_j - 9) + c\sum_{l=1}^{N} A_{jl}k_j^{-\beta}(z_l - z_j).
\end{aligned} \tag{4}$$

Finally, Kuramoto oscillators are represented by

$$\dot{\theta}_j = \omega_j + c\sum_{l=1}^{N} A_{jl}k_j^{-\beta}\sin(\theta_l - \theta_j) + \xi_j, \tag{5}$$

where θ_j and ω_j are the phase and frequency of j_{th} oscillator, respectively.

3. Results derived from Wang et al.

Wang et al. [14] proposed theoretical approaches for obtaining the functional relation between noisy fluctuation and node degree for the coupling matrix L, i.e., for the case of $\beta = 0$. In the following, we extend it for arbitrary β. Let us assume that the coupled dynamical systems in Eqs. (3)–(5) exhibit the synchronization without noise term ξ_j and the addition of noise does not drastically change the dynamical state. Therefore the dynamical variable of each node just fluctuates around the synchronous state, that is, mean trajectory over the ensemble $\langle x\rangle_E = \sum_{l=1}^{N} x_l/N$. Then the variational equation about the mean trajectory can be written by

$$\Delta\dot{x}_j \simeq -ck_j^{-\beta+1}\Delta x_j + ck_j^{-\beta}\sum_{l=1}^{N} A_{jl}\Delta x_l + \xi_j, \tag{6}$$

where $\Delta x_j = x_j - \langle x\rangle_E$. For the consensus dynamics, the sign of approximately equal must be replaced by that of equal. Otherwise, Eq. (6) is an approximated result. For Kuramoto oscillators, Δx_j denotes $\Delta\theta_j$ where $\Delta\theta_j = \theta_j - \langle\theta\rangle_E$.

First approximation for obtaining the functional relation is to neglect the second term in right-hand side of Eq. (6). Because the summation of Δx_l is negligible comparing to the Gaussian noise ξ_j [14]. In doing so, we obtain a linear stochastic differential equation and the power spectral density (PSD) of Δx_j is represented by

$$S_{\Delta x_j}(f) = \left|\frac{1}{2\pi if + ck_j^{-\beta+1}}\right|^2 S_{\xi_j}(f) = \frac{\sigma^2}{4\pi^2 f^2 + c^2k_j^{-2\beta+2}}. \tag{7}$$

From the result of Eq. (7), we can approximate the PSD of neglected term ($\chi_j := ck_j^{-\beta}\sum_{l=1}^{N} A_{jl}\Delta x_l$) in Eq. (6) by assuming that Δx_l are independent. By approximating $k_l^{-2\beta+2}$ in summation term to $\langle k\rangle^{-2\beta+2}$, the PSD of χ_j is described as

$$S_{\chi_j}(f) \simeq c^2k_j^{-2\beta}\sum_{l=1}^{N} A_{jl}S_{\Delta x_l}(f) \simeq \frac{c^2k_j^{-2\beta+1}\sigma^2}{4\pi^2f^2 + c^2\langle k\rangle^{-2\beta+2}}. \tag{8}$$

Now let us return to Eq. (6). It can be rewritten by

$$\Delta\dot{x}_j + ck_j^{-\beta+1}\Delta x_j = \chi_j + \xi_j. \tag{9}$$