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On the pumping effect in a pipe/tank flow configuration with friction

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HIGHLIGHTS

- We model the flow of a fluid in a horizontal pipe connected to a vertical tank.
- We deal with a singular second order ODE with periodic boundary conditions.
- Stable periodic solutions are obtained for some values of the physical parameters.

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ABSTRACT

We provide sufficient conditions for the existence and the asymptotic stability of periodic positive solutions for a pipe/tank flow configuration. The model is a nonlinear ordinary second order differential equation with a singularity containing the second power of the derivative of the unknown function and with a linear term representing friction. In this way we complement previous results obtained by G. Propst in 2006.

1. Introduction

A periodically forced differential system is called a *pump* if it has an asymptotically stable periodic solution with a non-equilibrium mean. A precise general definition of a pump may be found in [1] (see Definition 1). In the special case, when the system is governed by a second order scalar differential equation and we require the existence of a T-periodic solution, this definition reads as follows.

Notation 1.1. For a given continuous function $h : [0, T] \to \mathbb{R}$, we denote

$$\bar{h} = \frac{1}{T}\int_0^T h(s)\,\mathrm{d}s, \qquad h_* = \min\{h(t) : t \in [0, T]\} \quad \text{and}$$
$$h^* = \max\{h(t) : t \in [0, T]\}.$$

Definition 1.2. Let $T > 0$, $g : \mathbb{R}^3 \to \mathbb{R}$ and let $e : \mathbb{R} \to \mathbb{R}$ be a nonconstant T-periodic function, then we say that the equation

$$x'' = g(x, x', e(t)) \tag{1.1}$$

is a *periodically forced pump* if it has a T-periodic solution x such that

$$g(\bar{x}, 0, \bar{e}) \neq 0,$$

i.e. the mean value $\bar{x}$ of x is not an equilibrium of

$$x'' = g(x, x', \bar{e}).$$

Recently, G. Propst [1] presented an explanation of the pumping effect for flow configurations of 1–3 rigid tanks that are connected by rigid pipes. Moreover, he proved the existence of periodic solutions to the corresponding differential equations for systems of 2 or 3 tanks, while for the apparently simplest configuration consisting of 1 pipe and 1 tank, he guaranteed the existence of a positive periodic solution only in some particular cases.

Remark 1.3. The flow configurations considered by G. Propst in [1] are special cases of valveless systems of a moving fluid. The term

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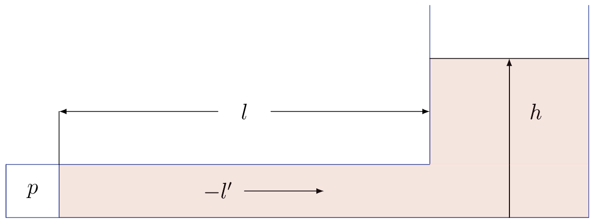

Fig. 1. 1 pipe–1 tank configuration.

valveless pumping refers to the conveyance of liquid fluids in mechanical systems that have no valves to ensure the preferential direction of flow. Such phenomena appear e.g. in the models of blood circulation in the cardiovascular system, in some other models from microfluidics, or, at large scales, in oceanic currents. Valveless pumping is also referred to as Liebau phenomenon, after the pioneering works by G. Liebau starting in 1954; see [2–4].

In the configuration of 1 pipe and 1 tank (see Fig. 1), a horizontal pipe is connected to a vertical tank, both contain a fluid of density ρ, up to the level height h in the tank. In the pipe, at distance ℓ from the tank, the pressure outside the mass- and frictionless moveable piston is forced T-periodically in time t, while the environmental pressure above the fluid in the open tank is set to zero. The time derivative of the momentum of the mass of the fluid in the pipe between the piston and the tank equals the sum of the forces acting on it. These forces are due to the pressure p and the hydrostatic pressure $\rho \cdot g \cdot h$ at the bottom of the tank, where g is the acceleration of gravity. The friction is modeled by Poiseuille's law with friction coefficient r_0. Furthermore, since the cross section A_P of the pipe is assumed to be small in comparison to the cross section A_T of the tank, the fluid in the tank is modeled to be at rest. Finally, by Bernoulli's equation, the pressure loss is due to the difference of the fluid velocity in the pipe and zero velocity in the tank, i.e., is given by $\zeta \frac{\rho}{2} w^2$, where $\zeta \geq 1$ is the junction coefficient depending on the particular geometry and smoothness of the junction of the tank and the pipe and $w = -\ell'$ is the fluid velocity in the pipe (oriented in the direction from the piston to the tank). Therefore, the fluid motion in the pipe is described by the momentum equation

$$\rho\,(\ell(t)\,w(t))' = p(t) - r_0\,\ell(t)\,w(t) - \rho\,g\,h(t) + \zeta\frac{\rho}{2}\,(w(t))^2. \tag{1.2}$$

The level height h is coupled to ℓ by the relation

$$A_P\,\ell(t) + A_T\,h(t) = V_0, \tag{1.3}$$

where the total volume V_0 of the fluid in the system is supposed to be constant. Inserting $w = -\ell'$ and (1.3) into (1.2), we get

$$\ell''(t) = -\frac{r_0}{\rho}\,\ell'(t) + \frac{1}{\ell(t)}\left(-\left(1+\frac{\zeta}{2}\right)(\ell'(t))^2 + \frac{g\,V_0}{A_T} - \frac{p(t)}{\rho}\right) - \frac{gA_P}{A_T}. \tag{1.4}$$

From the physical point of view we are interested in the search of positive solutions of problem (1.4). The right-hand side of the differential equation in (1.4) is singular for $\ell = 0$ and, moreover, the singular term

$$\frac{1}{\ell(t)}\left(-\left(1+\frac{\zeta}{2}\right)(\ell'(t))^2\right)$$

involves also the second power of the derivative ℓ' of ℓ. As there is lack of general existence results for such problems, this makes the analysis of the given model rather difficult. To our knowledge, only Hakl, Torres and Zamora considered in [5,6] remotely resembling problems for equations of the form

$$u'' + f(u)\,u' + g(t,u) = h(t,u),$$

where both f and g can have a singularity at the origin. Our aim is to fill the gap providing existence and stability results for the quite general periodic forcing p.

For the sake of clarity, let us denote

$$a = \frac{r_0}{\rho}, \qquad b = 1+\frac{\zeta}{2}, \qquad c = \frac{gA_P}{A_T}, \qquad e(t) = \frac{g\,V_0}{A_T} - \frac{p(t)}{\rho}, \quad \text{and} \quad u = \ell. \tag{1.5}$$

Then the given problem can be reformulated as the periodic boundary value problem

$$u'' + a\,u' = \frac{1}{u}\left(e(t) - b\,(u')^2\right) - c, \tag{1.6}$$

$$u(0) = u(T), \qquad u'(0) = u'(T). \tag{1.7}$$

According to the physical meaning of the involved parameters, we may assume

$$a \geq 0, \qquad b > 1, \qquad c > 0, \quad \text{and} \quad e \text{ is continuous and } T\text{-periodic on } \mathbb{R}. \tag{1.8}$$

Remark 1.4. The case $a = 0$ means that there is no Poiseuille friction, which might be considered an idealization but not meaningless from the modeling point of view. However, for the application of the theory presented in Section 2 we need to have a positive coefficient at the linear first order term of the second order boundary value problem, that is $a > 0$.

Assuming that u is a positive solution to (1.6), (1.7), multiplying (1.6) by $u(t)$ and integrating over the interval $[0, T]$, we get that the relation

$$\bar{e} - c\,\bar{u} = (b-1)\overline{(u')^2} \tag{1.9}$$

must hold. This yields immediately the following necessary condition for the existence of a positive solution to (1.6), (1.7).

Theorem 1.5. *Let (1.8) hold and let problem (1.6), (1.7) have a positive solution. Then*

$$\bar{e} > 0. \tag{1.10}$$

Remark 1.6. By (1.5), condition (1.10) is satisfied if and only if

$$\bar{p} < \rho\,g\,\frac{V_0}{A_T}. \tag{1.11}$$

This means that, if A_T and V_0 are fixed and the periodic forcing p has a positive mean value $\bar{p}$, then problem (1.6), (1.7) has no solution whenever the fluid density ρ is small enough.

Furthermore, put

$$g(x,y,z) = -a\,y + \frac{1}{x}\left(z - b\,y^2\right) - c \quad \text{for } x \in (0,\infty),\ y,z \in \mathbb{R}.$$

Then Eq. (1.6) takes the form (1.1). Let u be an arbitrary positive nonconstant solution of (1.6), (1.7). By (1.8) and (1.9) we have

$$g(\bar{u}, 0, \bar{e}) = \frac{1}{\bar{u}}\,(\bar{e} - c\,\bar{u}) = \frac{1}{\bar{u}}\,(b-1)\overline{(u')^2} > 0,$$

i.e. $x = \bar{u}$ is not a zero of $g(x, 0, \bar{e})$. Therefore, the following simple observation is true, as well.