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Further understanding of Huygens' coupled clocks: The effect of stiffness

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HIGHLIGHTS

- The classical Huygens' experiment on synchronization of clocks is revisited.
- The influence of the stiffness on the onset of synchronization is investigated.
- A perturbation method is used to derive conditions for synchronization.
- We show that the in-phase solution is strongly affected by the stiffness.

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ABSTRACT

A simplified model of the classical Huygens' experiment on synchronization of pendulum clocks is examined. The model consists of two pendula coupled by an elastically supported rigid bar. The synchronized limit behaviour of the system, i.e. in-phase and anti-phase synchronization of the pendula, is studied as a function of the stiffness of the spring that supports the coupling bar. It is demonstrated that the stiffness has a large influence on the existence, stability, and oscillation frequency of the in-phase solution. The relationship between the obtained results and experimental results that have been reported in the literature, including Huygens' original observations, is stressed.

1. Introduction

On a quiet day in late February 1665, the Dutch scientist Christiaan Huygens made a serendipitous discovery while being confined to his bedroom for several days due to an illness: two pendulum clocks hanging from a common support, a wooden bar supported by two chairs (see Fig. 1(a)), kept in pace relative to each other such that the two pendula always swung together (in opposite motion) and never varied. This was called by Huygens "the sympathy of two clocks". By performing some systematic experiments he found the source of the "sympathy", namely the imperceptible motion of the bar to which the pendulum clocks were attached [1–3]. After Huygens' observations, other authors reported similar experiments with pendulum clocks [4–6]. Earlier attempts to provide a theoretical explanation of the sympathy of clocks described by Huygens are presented in [3,7].

During the last decade, the study of Huygens' synchronization, i.e. synchronization of oscillators (e.g. pendula) coupled by a suspended platform, has regained attention. Several authors have contributed towards a better understanding of the phenomenon. We can distinguish two approaches: a theoretical approach and one oriented to obtain insight by experimental analysis. Some works related to the theoretical approach are presented in [8–18]. In particular, in [17] a fairly complete study of the synchronous motion of pendulum clocks is presented. The study is conducted by considering the energy balance in the system and showing how the energy is transferred between the pendula via the oscillating beam. Moreover, by means of computer simulations and experiments, the influence of certain parameter values, including mass and length of the pendula, on the occurrence of synchronous limit solutions, has been investigated. In the present contribution, however, we will restrict the analysis to a single parameter, namely the stiffness of the coupling bar, and derive analytic conditions for the existence, stability, and oscillation frequency of the synchronous solutions in terms of the stiffness parameter.

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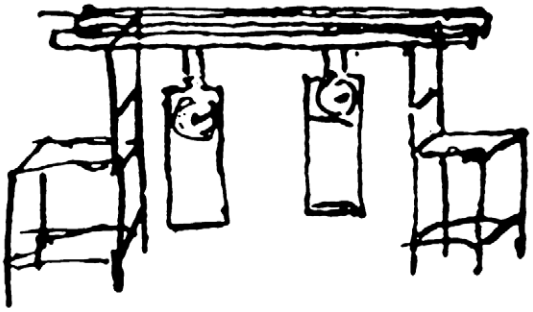
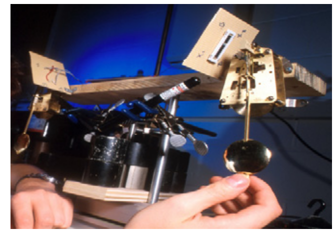
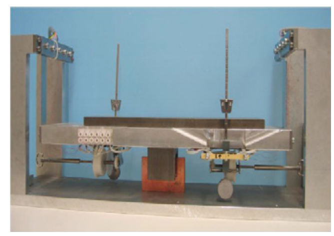


(a) Huygens' setup [2].



(b) Wiesenfeld's setup [20].

(c) Pantaleone's setup [19].



(d) Nijmeijer's setup [21].

Fig. 1. Original Huygens' setup and its modern versions.

For the second approach several experimental setups have been created by researchers in order to reproduce the observations made by Huygens and find mathematical arguments for the synchronized motion of the clocks. The most representative setups (see Fig. 1) are presented in [19–21], all aiming at demonstrating that in-phase or anti-phase synchronization of a number of pendulum-like oscillators (metronomes) can be achieved. In these setups, besides the synchronized motion, other kinds of motion have been reported, as for example, beating death, where one of the metronomes stops its oscillations, whereas the other one keeps oscillating [20], and intermediate synchronization, i.e. metronomes synchronize with constant phase difference but different amplitude [21].

Besides the aforementioned efforts, there exist unanswered questions regarding Huygens' synchronization. For instance, consider a pair of metronomes placed on top of a suspended platform. If the metronomes synchronize (in-phase or in anti-phase), what is then the oscillation frequency of the synchronous behaviour? Is it the average of the individual eigenfrequencies of the metronomes? Or is the oscillation frequency determined by the 'fastest' metronome, or by the 'slowest' metronome? Moreover, how do the parameters of the coupling influence the frequency of the synchronized limit solution? If these questions are analysed by using Pantaleone's setup [19], then the reader may conclude that the oscillation frequency is greater than the average of the individual eigenfrequencies (see for example the experiments reported in [22]). However, if Nijmeijer's setup is used, then the reader may come to the conclusion that the oscillation frequency is lower than the average of the individual eigenfrequencies (see the experiments reported in [21]). Hence, an obvious question is, what is the reason behind this 'discrepancy'? Moreover, what would be the answer if the original Huygens' setup is used?

The purpose of this paper is to provide a further understanding of Huygens' synchronization problem. In the analysis, a simplified model consisting of two pendula mounted on a rigid bar, which is horizontally connected to a fixed support by means of a spring, is considered. The influence of the stiffness of the spring in the onset of in-phase and anti-phase synchronization in the coupled pendula is analytically and numerically examined. Note that a similar analysis has been conducted in [16]. In that study, however, the damping of the platform has been considered as the key parameter.

The outcome of our analysis reveals that the stiffness has a large influence not only on the oscillation frequency of the in-phase solution, but also on its existence and stability properties. On the other hand, for the anti-phase solution these properties are independent of the stiffness in the coupling although the domain of attraction of the anti-phase solution will change if the in-phase solution coexists.

The manuscript is organized as follows. The model and framework for analysis are described in Section 2. The case of identical pendula is discussed in Section 3, whereas in Section 4 the study is extended to the case of nonidentical pendula. In the former case, an analytic study is conducted by using a result based on the Poincaré method of perturbation and the obtained results are supported by computer simulations. For the nonidentical pendula, since an analytic study turns out to be very involved, a numerical analysis is conducted. Finally, in Section 5, a discussion about the obtained results is presented and it is also explained why Pantaleone's setup and Nijmeijer's setup show different experimental results.

2. Model and strategy

This section introduces the model used in the analysis and a mathematical framework that is suitable for studying Huygens' synchronization. The framework, which was introduced by Blekhman [23], allows to derive conditions for the existence and stability of synchronous solutions and to determine the oscillation frequency of these solutions. The strategy of analysis presented in this section is used later (see Section 3) to demonstrate that the stiffness in the coupling influences the synchronous behaviour in the system.

2.1. Model

A simplified Huygens' system depicted in Fig. 2 is considered.

This model consists of a coupling bar of mass M [kg], which is elastically supported via a linear spring with elasticity k [N/m] and a linear viscous damper with damping constant b [Ns/m]. Pendulum i is modelled by a point mass of mass m_i [kg] attached at the lower end of a massless bar of length l_i [m] for $i = 1, 2$. The