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Renormalisation of correlations in a barrier billiard: Quadratic irrational trajectories

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HIGHLIGHTS

- We study correlations in barrier billiards at quadratic irrational frequencies.
- The autocorrelation function (ACF) may display self-similar or chaotic behaviour.
- We calculate the asymptotic height & location of peaks in the ACF for the half barrier.
- Chaotic behaviour of the ACF is shown to occur for more general barriers.
- We show the presence of invariant sets representative of the underlying dynamics.

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ABSTRACT

We present an analysis of autocorrelation functions in symmetric barrier billiards using a renormalisation approach for quadratic irrational trajectories. Depending on the nature of the barrier, this leads to either self-similar or chaotic behaviour. In the self-similar case we give an analysis of the half barrier and present a detailed calculation of the locations, asymptotic heights and signs of the main peaks in the autocorrelation function. Then we consider arbitrary barriers, illustrating that typically these give rise to chaotic correlations of the autocorrelation function which we further represent by showing the invariant sets associated with these correlations. Our main ingredient here is a functional recurrence which has been previously derived and used in work on the Harper equation, strange non-chaotic attractors and a quasi-periodically forced two-level system.

1. Introduction

Barrier billiards are a class of dynamical system with autocorrelation functions which have been shown to possess self-similar behaviour. The topic was first motivated by Wiersig [1] who showed that the system could be reduced to a skew-product evolution equation based on the sign of the phase space variable. The key result of Wiersig was that phase space functions for this system exhibit singular continuous spectra. This work also provided evidence that the associated autocorrelation function never decays to 0 or returns to 1.

This result was explored in [2] and it was discovered that for the golden mean frequency, in the case of the so-called half barrier, the autocorrelation function (ACF) has self similar form at Fibonacci times. It was shown that the ACF has peaks of magnitude $1-1/\sqrt{5}$ at every third Fibonacci number time, whereas it is zero at all other such times. As in previous studies relating to the birth and ACFs of strange nonchaotic attractors [3–6] in addition to analysis of self-similar fluctuations of localised eigenstates of the Harper equation [7–9], key to this analysis is the functional recurrence

$$Q_n(x) = Q_{n-1}(-\omega x)Q_{n-2}(\omega^2 x + \omega). \quad (1.1)$$

The limitation of this work is that it only deals with the golden mean frequency case, and the purpose of this article is to widen these results to a more general class of quadratic irrational frequencies, namely those of the form

$$\omega = \frac{\sqrt{m^2+4}-m}{2}. \quad (1.2)$$

These are the numbers whose continued fraction representation is $[m, m, \ldots]$.

The nature of this generalisation is similar to work completed by the second author and his colleague on correlation properties of a quasi-periodically forced two-level system in [10]. This paper built upon the theory previously developed in [11], which focused on the golden mean forcing frequency, and extended the results to quadratic irrationals of this form. The derived functional recurrence in the golden mean case is the additive version of (1.1), and we find that the same is true in the quadratic irrational case. It is

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shown in [10] that for a particular choice of modulation function, this two-level system has an ACF with peaks of magnitude $1 - 1/\sqrt{m^2+4}$. Somewhat surprisingly it was discovered that there is a dichotomy between the cases m odd, and m even with regard to when these peaks occur.

Similarly, the strong coupling fixed point of the Harper equation and the generalised Harper equation have both been considered for quadratic irrationals of this form [12,13]. In [7] while working with the golden mean flux the authors discover a renormalisation strange set which they name "the orchid"; in [13] such a set is produced for each value of m giving rise to a "garden of orchids" (this point will be expanded upon in the conclusion).

In [14] we generalised our study of the barrier billiard problem for the golden mean frequency to arbitrary barriers, and showed that if the height of the barrier is chosen outside the field $\mathbb{Q}(\omega)$ the correlations at successive Fibonacci numbers are chaotic. The chaoticity of the time series of correlations at Fibonacci times was confirmed by the calculation of a positive largest Lyapunov exponent using Rosenstein's method [15]. Furthermore, using the well known embedding theorem of Takens [16] we reconstructed an embedding of the dynamics in three dimensional space, revealing the presence of an invariant set which we may call "the star".

In Section 2 we briefly summarise the equations of motion and derive the functional recurrence which plays the same role as (1.1) for quadratic irrationals. We see that (as expected) it is the multiplicative version of the recurrence found in [10]. Section 3 gives the definition of a map F whose periodic orbits correspond to discontinuities of our renormalised functions Q_n, and using symbolic dynamics we further deduce information about the period of Q_n based on initial discontinuity data. We summarise results from the golden mean frequency case studied in [2] which carry over to our study of quadratic irrational frequencies. In Section 4 we find necessary conditions for Q_n to be periodic and give a crucial formula for the discontinuity location sets. In Sections 5 and 6 we extend the work on the half barrier for the golden mean frequency covered in [2] to our class of quadratic irrational frequencies.

It is shown for the half-barrier that peaks of magnitude $1 - 1/\sqrt{m^2+4}$ occur in the ACF, and that the location of these peaks is determined by the parity of m. Furthermore there is a further distinction between the cases of $m \equiv 0 \pmod 4$ and $m \equiv 2 \pmod 4$; in the former the ACF only displays positive peaks meaning that there is strict correlation whereas in the latter these peaks alternate in sign which indicates "anticorrelation" which is also seen in the case of m odd.

Finally, in Section 7 we extend our previous work on chaotic correlations in barrier billiards at the golden mean frequency [14] to examples when the frequency is taken from this class of quadratic irrationals. We numerically approximate the autocorrelation function for the system at characteristic times and for each frequency studied we show the existence of invariant sets on which the correlations lie.

2. Derivation of the relevant equations and functional recurrence

To begin this section we summarise the derivation of the equations of motion as originally shown in [1].

The problem of symmetric barrier billiards relates to the motion of a point unit mass moving at constant velocity in the rectangular chamber $[-1, 1] \times [0, 1]$. The particle is experiencing elastic collisions with the boundary of this chamber according to the law the angle of incidence is equal to the angle of reflection (we ignore trajectories which hit corners). A vertical barrier is placed in the middle of this chamber. It is clear that the evolution of $(|x|, y)$ is simply the integrable evolution of the billiard in $[0, 1] \times [0, 1]$ which can be written in terms of action-angle variables (θ_x, θ_y).

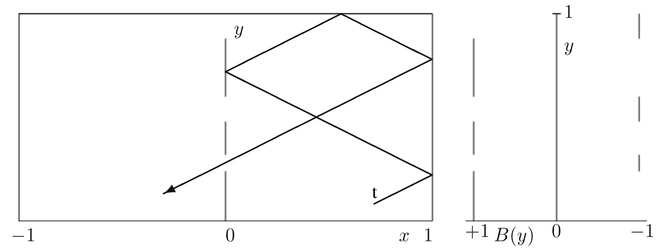

Fig. 1. A symmetric barrier billiard and its corresponding barrier function $B(y)$.

To convert to x–y coordinates we define $x = f(\theta_x)$, $y = f(\theta_y)$ where f is the tent map

$$f(\theta) = \begin{cases} 2\theta, & \theta \in [0, 1/2] \\ 2(1-\theta), & \theta \in [1/2, 1]. \end{cases} \tag{2.1}$$

Note that f is invertible if we adopt the understanding that $\theta \in [0, 1/2]$ corresponds to positive velocity.

The nature of the barrier is determined by a barrier function $B(y) : [0, 1] \to \{-1, 1\}$ which takes the value 1 if the barrier is present at y and -1 elsewhere. We ignore trajectories which hit endpoints of the pieces of the barrier. Fig. 1 shows a schematic of the system and its associated barrier function.

Defining

$$x(t) = s(t)|x(t)|, \tag{2.2}$$

where $s(t)$ is the sign of x at time t, the question of understanding the system boils down to understanding the evolution of s. The sign s can only change sign when $x(t) = 0$. Hence taking a stroboscopic sample $x = 0$ we repeatedly evaluate $s(t)$ just before the particle strikes the barrier. We define y_n be the value of y at time step n (the successive times when $x = 0$), and θ_n to be the corresponding value of θ_y.

If the barrier is absent at y_n then the sign of x will change, otherwise it will remain the same.

In summary we have reduced the problem of understanding symmetric barrier billiards to that of understanding the skew-product system

$$\theta_{n+1} = \theta_n + \omega \pmod 1, \tag{2.3}$$

$$s_{n+1} = \Phi(\theta_n) s_n. \tag{2.4}$$

We see that the solution of the system is given by

$$\theta_n = \theta_0 + n\omega \pmod 1, \tag{2.5}$$

$$s_n = s_0 \prod_{k=0}^{n-1} \Phi(\theta_0 + k\omega). \tag{2.6}$$

If ω is rational with then it is clear that s_n is periodic and we have a complete understanding of the dynamics. However we can gather no such information if ω is irrational and so we resort to analysing the autocorrelation function (ACF) defined as

$$C(t) = \langle s_n s_{n+t} \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} s_n s_{n+t}. \tag{2.7}$$

Using the same approach as in [3,2] we now use the ergodicity in θ to convert to a phase average:

$$C(t) = \int_0^1 S_t(\theta) d\theta,$$

where

$$S_t(\theta) = \prod_{k=0}^{t-1} \Phi(\theta + k\omega), \; t \ge 1, \qquad S_0(\theta) = 1. \tag{2.8}$$

Now we restrict our attention to quadratic irrational frequencies of the form

$$\omega = \frac{\sqrt{m^2+4} - m}{2}. \tag{2.9}$$