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The scattering map in two coupled piecewise-smooth systems, with numerical application to rocking blocks☆

A. Granados [a,*], S.J. Hogan [b], T.M. Seara [c]

[a] INRIA Paris—Rocquencourt Research Centre, Domaine de Voluceau, BP105, 78153 Le Chesnay cedex, France

[b] Department of Engineering Mathematics, University Walk, Bristol BS8 1TR, UK

[c] Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain

HIGHLIGHTS

- We prove the persistence of C^0 normally-hyperbolic manifolds.
- We extend Melnikov methods to show the persistence of C^0 heteroclinic manifolds.
- We formally define the scattering map.
- We obtain first order properties relevant for quantifying accumulated energy.
- We numerically verify the results in the model of two coupled rocking blocks.

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ABSTRACT

We consider a non-autonomous dynamical system formed by coupling two piecewise-smooth systems in $\mathbb{R}^2$ through a non-autonomous periodic perturbation. We study the dynamics around one of the heteroclinic orbits of one of the piecewise-smooth systems. In the unperturbed case, the system possesses two C^0 normally hyperbolic invariant manifolds of dimension two with a couple of three dimensional heteroclinic manifolds between them. These heteroclinic manifolds are foliated by heteroclinic connections between C^0 tori located at the same energy levels. By means of the *impact map* we prove the persistence of these objects under perturbation. In addition, we provide sufficient conditions of the existence of transversal heteroclinic intersections through the existence of simple zeros of Melnikov-like functions. The heteroclinic manifolds allow us to define the *scattering map*, which links asymptotic dynamics in the invariant manifolds through heteroclinic connections. First order properties of this map provide sufficient conditions for the asymptotic dynamics to be located in different energy levels in the perturbed invariant manifolds. Hence we have an essential tool for the construction of a heteroclinic skeleton which, when followed, can lead to the existence of Arnold diffusion: trajectories that, on large time scales, destabilize the system by further accumulating energy. We validate all the theoretical results with detailed numerical computations of a mechanical system with impacts, formed by the linkage of two rocking blocks with a spring.

1. Introduction

This paper is concerned with the question of whether it is possible to observe Arnold diffusion [1] in systems governed by piecewise-smooth differential equations, to which known results in the field cannot be directly applied. Arnold diffusion occurs when there is a large change in the action variables in nearly integrable Hamiltonian systems. Systems governed by piecewise-smooth differential equations are widespread in engineering, economics, electronics, ecology and biology; see [2] for a recent comprehensive survey of the field.

Action variables are conserved for integrable systems. When such systems are perturbed, for example, by a periodic forcing, KAM theory tells us that the value of these variables stays close to their conserved values for *most* solutions. Subsequently Arnold [1] gave an example of a nearly integrable system for which there was large growth in the action variables.

There has been a lot of activity in the field of Arnold diffusion in recent years and a large variety of results that have been obtained or announced. We refer to [3–6] for a detailed survey of recent

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* Corresponding author. Tel.: +33 139635258.
E-mail addresses: alberto.granados_corsellas@inria.fr, albert.granados@inria.fr (A. Granados), s.j.hogan@bristol.ac.uk (S.J. Hogan), tere.m-seara@upc.edu (T.M. Seara).

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results. Up to now, there are mainly two kind of methods used to prove the existence of instabilities in Hamiltonian systems close to integrable; variational methods [7–15] and the so-called geometric methods [16–22], both of which have been used to prove generic results or study concrete examples. Other approaches us both methods and functional analysis techniques, see for example [23].

The study of Arnold diffusion using geometric methods has been greatly facilitated by the introduction [16–18] of the *scattering map* of a normally hyperbolic invariant manifold with intersecting stable and unstable invariant manifolds along a homoclinic manifold. This map finds the asymptotic orbit in the future, given an asymptotic orbit in the past. Perturbation theory of the scattering map [18] generalizes and extends several results obtained using Melnikov's method [24,25].

For planar regular systems under non-autonomous periodic perturbations, Melnikov's method is used to determine the persistence of periodic orbits and homoclinic/heteroclinic connections by guaranteeing the existence of simple zeros of the subharmonic Melnikov function and the Melnikov function, respectively. The main idea is to consider a section normal to the unperturbed vector field at some point on the unperturbed homoclinic/heteroclinic connection. Then it is possible to measure the distance between the perturbed manifolds, owing to the regularity properties of the stable and unstable manifolds of hyperbolic critical points in smooth systems.

In [26] these classical results were rigorously extended to a general class of piecewise-smooth differential equations, allowing for a general periodic Hamiltonian perturbation, with no symmetry assumptions. For such systems, the unperturbed system is defined in two domains, separated by a *switching manifold* Σ, each possessing one hyperbolic critical point either side of Σ. In this case, the vector normal to the unperturbed vector field is not defined everywhere. By looking for the intersection between the stable and unstable manifolds with the switching manifold, an asymptotic formula for the distance between the manifolds was obtained. This turned out to be a *modified* Melnikov function, whose zeros give rise to the existence of heteroclinic connections for the perturbed system. The general results in [26] were then applied to the case of the rocking block [27,28] and excellent agreement was obtained with the results of [28].

Following these ideas, in this paper we study a system which consists of a non-autonomous periodic perturbation of a piecewise-smooth integrable Hamiltonian system in $\mathbb{R}^4$. The unperturbed system is given by the product of two piecewise-smooth systems. We assume that one of them has two hyperbolic critical points of saddle type with a pair of C^0 heteroclinic orbits between them. The other system behaves as a classical integrable system with a region foliated by C^0 periodic orbits. Therefore, the product system looks like a classical *a priori* unstable Hamiltonian system [29], possessing two C^0 normally hyperbolic invariant manifolds of dimension two with a couple of three dimensional C^0 heteroclinic manifolds.

The main difficulty in following the program of [17] is that we couple two piecewise-smooth systems, each of which possesses its own switching manifold. Therefore, when considering the product system, we need to deal with a piecewise-smooth system in $\mathbb{R}^4$ with two 3-dimensional switching manifolds that cross in a 2-dimensional one. Therefore the classical impact map associated with one switching manifold will be piecewise-smooth in general. We overcome this difficulty by restricting the impact map to suitable domains so that we can apply classical results for normally hyperbolic invariant manifolds and their persistence and obtain a scattering map between them with explicit asymptotic formulae.

Note that, in this paper, we restrict our attention to the study of the scattering map and we do not rigorously prove the existence of Arnold diffusion. Due to the continuous nature of the system considered in this paper, the method of correctly aligned windows [19] seems to be very suitable for application to our model for this purpose. In fact, recent results in [30], which do not rely on the use of KAM theory, appear to be capable of extension to piecewise-smooth systems in order to achieve this goal.

Piecewise-smooth systems are found in a host of applications [2]. A simple example is the rocking block model [27], which has wide application in earthquake engineering and robotics. This piecewise-smooth system has been shown to possess a vast array of solutions [28]. The model has been extended to include, for example, stacked rigid blocks [31] and multi-block structures [32]. Particular attention is paid to the case of block overturning in the presence of an earthquake, as this has consequences for safety in the nuclear industry [33] and for the preservation of ancient statues [34]. Within the context of the current paper, Arnold diffusion could be seen as one possible mechanism for block overturning, when the perturbation (earthquake) of an apparently stable system (two blocks coupled by a simple spring) leads to overturning. An early application of Melnikov theory to the rocking block problem [35] involved the calculation of the *stochastic* Melnikov criterion of instability for a multidimensional rocking structure subjected to *random* excitation.

Note that we are considering the class of piecewise-smooth differential equations that involve *crossing* [2], where the normal components of the vector field either side of the switching manifold are in the *same* sense. When these components are in the *opposite* sense, *sliding* can occur [2]. The extension of the Melnikov method to this case is still in its infancy [36].

The paper is organized as follows. In Section 2 we present the system we will consider and the main piecewise-smooth invariant geometrical objects that will play a role in the process. In Section 3 we present the impact map associated with one switching manifold in the extended phase space and its domains of regularity and provide an explicit expression for it in the unperturbed case. In Section 4 we study some regular normally hyperbolic invariant manifolds for the impact map which correspond to the piecewise-smooth ones for the flow in the extended phase space. We then apply classical perturbation theory to demonstrate the persistence of the normally hyperbolic invariant manifolds and their stable and unstable manifolds and deduce the persistence of the corresponding invariant manifolds for the perturbed flow. This allows us to give explicit conditions for the existence of transversal heteroclinic manifolds in the perturbed system in terms of a modified Melnikov function and to derive explicit formulae for the scattering map in Section 5. In particular, we obtain formulae for the change in the energy of the points related by the scattering map and in the average energy along their orbits. In Section 6 we illustrate the theoretical results of Section 5 with numerical computations for two coupled rocking blocks subjected to a small periodic forcing. We use the simple zeros of the Melnikov function to numerically compute heteroclinic connections linking, forwards and backwards in time, two trajectories at the invariant manifolds. These trajectories correspond to one block performing small rocking oscillations while the other block rocks about one of its heteroclinic orbits. During this large, fast, excursion, the amplitude of the rocking block oscillations may lead to an increase or decrease in its average energy. Using the first order analysis of the scattering map we are able to approximately predict the magnitude of this change, which is in excellent agreement with our numerical computations.

2. System description

2.1. Two uncoupled systems

In this paper we consider a non-autonomous dynamical system formed by coupling two piecewise-smooth systems in $\mathbb{R}^2$ through