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Stability of stationary solutions for nonintegrable peakon equations

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HIGHLIGHTS

- We consider a family of equations which generalize the Camassa–Holm equation.
- The equations of this family are not integrable but do admit peakon solutions.
- Numerical studies reported by Holm and Staley indicate changes in the stability of the Peakon solutions.
- We describe analytical results on one of these bifurcation phenomena.
- We show that in a suitable parameter range there are stationary solutions which are orbitally stable.

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ABSTRACT

The Camassa–Holm equation with linear dispersion was originally derived as an asymptotic equation in shallow water wave theory. Among its many interesting mathematical properties, which include complete integrability, perhaps the most striking is the fact that in the case where linear dispersion is absent it admits weak multi-soliton solutions – "peakons" – with a peaked shape corresponding to a discontinuous first derivative. There is a one-parameter family of generalized Camassa–Holm equations, most of which are not integrable, but which all admit peakon solutions. Numerical studies reported by Holm and Staley indicate changes in the stability of these and other solutions as the parameter varies through the family.

In this article, we describe analytical results on one of these bifurcation phenomena, showing that in a suitable parameter range there are stationary solutions – "leftons" – which are orbitally stable.

1. Introduction

The family of partial differential equations

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}, \qquad (1)$$

labelled by the parameter b, is distinguished by the fact that it includes two completely integrable equations, namely the Camassa–Holm equation (the case $b = 2$ [1,2]), and the Degasperis–Procesi equation (the case $b = 3$ [3,4]). Each of the two integrable equations arises as the compatibility condition for an associated pair of linear equations (a Lax pair), and the latter leads to other hallmarks of integrability, namely the inverse scattering transform, multi-soliton solutions [5,6,2], an infinite number of conservation laws, and a bi-Hamiltonian structure. (The latter structure for the case $b = 2$ was found in [7].) According to various tests for integrability, the cases $b = 2, 3$ are the only integrable equations within this family [3,8–10].

The Camassa–Holm equation was originally proposed as a model for shallow water waves [1,2], and it is explained in [11,12] that the members of the family of Eqs. (1), apart from the case $b = -1$, are asymptotically equivalent by means of an appropriate Kodama transformation. The results of [13] (see Proposition 2 therein, and also Eq. (3.8) in [14]) show that, in a model of shallow water flowing over a flat bed, the solution u of (1) corresponds to the horizontal component of velocity evaluated at the level line (fraction of the total depth measured from the bottom) $\theta \in [0, 1]$, where $\theta = \sqrt{\frac{11b-10}{12b}}$, which requires either $b \geq 10/11$ or $b \leq -10$. However, there continues to be debate in the literature about the precise range of validity of such models [15].

Another aspect of Eqs. (1) that makes them the focus of much interest is the special solutions that they admit. Although (as already mentioned) there are multi-soliton solutions for $b = 2, 3$, these smooth solutions only exist on a zero background in the case where the equation has additional linear dispersion terms (terms proportional to u_x and/or u_{xxx}, that is); such terms can be removed by a combination of a Galilean transformation together with a shift

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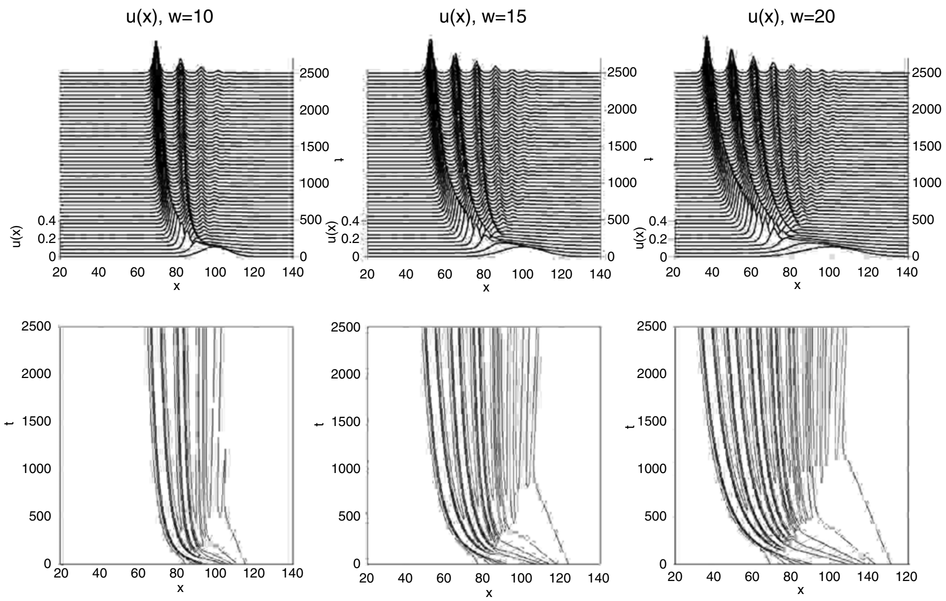

Fig. 1. Leftons evolve from Gaussian initial profiles of different widths in the case $b = -3$.
Source: Reproduced with kind permission from [20].

$u \to u + u_0$, which (for $u_0 \neq 0$) changes the boundary conditions at spatial infinity. In the case of vanishing boundary conditions at infinity, there are no smooth multi-soliton solutions, but Camassa and Holm noticed that for $b = 2$ and any positive integer N there are instead weak solutions given by

$$u(x,t) = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|}, \tag{2}$$

which have the form of a linear superposition of N peaked waves whose positions q_j and amplitudes p_j are respectively the canonically conjugate coordinates and momenta in a finite-dimensional Hamiltonian system that is completely integrable in the Liouville–Arnold sense. When $b = 2$, Hamilton's equations correspond to the geodesic equations for an N-dimensional manifold with coordinates $q_1, \dots, q_N$ and co-metric $g^{ij} = e^{-|q_i-q_j|}$. The form of the multi-peakon solutions (2) persists for all values of b, although in general the Hamiltonian system governing the time evolution of the positions and amplitudes is non-canonical [16], and for $N > 2$ this finite-dimensional dynamics is expected to be integrable only when $b = 2, 3$.

In the case $b = 2$, it is known that the Camassa–Holm equation is of Euler–Poincaré type, corresponding to a geodesic flow with respect to the H^1 metric on a suitable diffeomorphism group [17]; the geodesic equations for the N-peakon solutions (2) are a finite-dimensional reduction of this flow [18]. Although the standard geodesic interpretation, in terms of a metric, is lost for other values of b, it was recently shown that the periodic case of the Degasperis–Procesi and the other equations in the b-family can be regarded as geodesic equations for a non-metric connection on the diffeomorphism group of the circle [19].

Holm and Staley made an extensive numerical study of solutions of (1) for different values of b, by starting with different initial profiles and observing how they evolved with time and with changing b [21,20]. They observed that, broadly speaking, there are three distinct parameter regimes with quite different behaviour, separated by bifurcations at $b = 1$ and $b = -1$, as follows:

Peakon regime: For $b > 1$, arbitrary initial data asymptotically separates out into a number of peakons as $t \to \infty$.

Ramp-cliff regime: For $-1 < b < 1$, solutions behave asymptotically like a combination of a "ramp"-like solution of Burgers equation (proportional to x/t), together with an exponentially-decaying tail ("cliff").

Lefton regime: For $b < -1$, arbitrary initial data moves to the left and asymptotically separates out into a number of "leftons" as $t \to \infty$, which are smooth stationary solitary waves. (See Fig. 1.)

The behaviour observed separately in each of the parameter ranges $b > 1$ and $b < -1$ can be understood as particular instances of the *soliton resolution conjecture* [22], a vaguely defined conjecture which states that for suitable dispersive wave equations, solutions with "generic" initial data will decompose into a finite number of solitary waves plus a radiation part which decays to zero. In this article, our aim is to provide a first step towards explaining this phenomenon analytically for Eq. (1) in the "lefton" regime $b < -1$. We show that in this parameter range a single lefton solution is orbitally stable, by applying the approach of Grillakis, Shatah and Strauss in [23]. The main ingredients required for our stability analysis are the Hamiltonian structure and conservation laws for (1). The lefton solutions are a critical point for a functional which is combination of the Hamiltonian and a Casimir, but the second variation has some negative spectrum, so it is not possible to apply the energy-Casimir method as in [24].

In the next section we describe the Hamiltonian structure and conservation laws of (3) that exist for all b. After that we consider orbital stability of stationary waves when $b < -1$: see Theorem 2 in Section 3 for the main result of the paper. We make some remarks about other values of b in our conclusions.

2. Conserved quantities and Hamiltonian structure

In order to better understand the properties of each equation in the family (1), it is convenient to rewrite it in the following way:

$$m_t + um_x + bu_x m = 0, \quad m = u - u_{xx}. \tag{3}$$

This can be regarded as a nonlocal evolution equation for m, where (at each time t) the field u is obtained from m by the convolution

$$u(x) = g * m(x) = \int_{-\infty}^{\infty} g(x-y)m(y)dy, \qquad g(x) = \frac{1}{2}\exp(-|x|). \tag{4}$$

Henceforth we use the symbol $\int$ without limits to denote an integral $\int_{\mathbb{R}}$ over the whole real line.

From the Eq. (1) written in the nonlocal form (3) it is straightforward to verify that, for any value of $b \neq 0, 1$, there are at least