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Invariant parameterization and turbulence modeling on the beta-plane

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HIGHLIGHTS

- Complete description of differential invariants for vorticity equation on beta-plane.
- Invariant parameterization by moving frame invariantization.
- Invariant hyperdiffusion for the simulation of two-dimensional turbulence.
- Combination of conservative and invariant parameterizations is proposed.

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ABSTRACT

Invariant parameterization schemes for the eddy-vorticity flux in the barotropic vorticity equation on the beta-plane are constructed and then applied to turbulence modeling. This construction is realized by the exhaustive description of differential invariants for the maximal Lie invariance pseudogroup of this equation using the method of moving frames, which includes finding functional bases of differential invariants of arbitrary order, a minimal generating set of differential invariants and a basis of operators of invariant differentiation in an explicit form. Special attention is paid to the problem of two-dimensional turbulence on the beta-plane. It is shown that classical hyperdiffusion as used to initiate the energy–enstrophy cascades violates the symmetries of the vorticity equation. Invariant but nonlinear hyperdiffusion-like terms of new types are introduced and then used in the course of numerically integrating the vorticity equation and carrying out freely decaying turbulence tests. It is found that the invariant hyperdiffusion scheme is closely reproducing the theoretically predicted k^{-1} shape of enstrophy spectrum in the enstrophy inertial range. By presenting conservative invariant hyperdiffusion terms, we also demonstrate that the concepts of invariant and conservative parameterizations are consistent.

1. Introduction

As atmospheric and oceanic numerical models get increasingly complex, it becomes more and more challenging to propose valuable conceptual paradigms for those processes that the model is still not able to capture owing to its limited spatial and temporal resolution. This problem is common to all numerical models irrespective of their eventual degree of sophistication [1,2]. In the beginning of numerical modeling in geophysical fluid dynamics, it was often the lack of computer power that dictated which processes had to be parameterized, even with a concise understanding of these processes. As computers became more capable, the problem of parameterization shifted to processes occurring on rather fine scales where it can be difficult to retrieve accurate experimental data. Accordingly, for various processes that should be taken into account in order to improve the forecast range of a numerical model, there is still no satisfactory general understanding. This naturally makes it difficult to set up valuable parameterization schemes, which for this reason is usually an elaborate task.

On the other hand, processes that occur in geophysical fluid dynamics and that can be described using differential equations also might have certain structural or geometrical properties. Such properties can be conservation of mass or energy or other fundamental conservation laws. Real-world processes are generally also invariant under specific transformation groups, as e.g. the Galilean group. This is why one can ask the question whether it is reasonable to construct parameterization schemes

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for processes possessing certain structural features in a manner such that these features are preserved in the closed model. In this way, even if a model is not able to explicitly resolve processes, loosely speaking, it takes into account some of their significant properties. This study was initiated in [3] for the problem of finding invariant turbulence closure schemes for the filtered Navier–Stokes equations. In the present paper we aim to give a further instance for invariant parameterization schemes by constructing closure ansatzes that retain certain Lie point symmetries of the barotropic vorticity equation on the beta-plane.

This possible stream of constructing geometrically motivated parameterization schemes in some sense parallels the present general trend in numerical modeling to design specially adapted discretizations of differential equations that capture a range of their qualitative or global features, such as conservation laws, a Hamiltonian structure or symmetry properties. Especially the possibility of constructing discretization schemes that have the same symmetries and/or conservation laws as the original differential equations they are a model of, as proposed and discussed e.g. in [4–11], is of immediate relevance to the present work. This is because, strictly speaking, a discretization of a system of differential equation is in practice not enough to set up a valuable numerical model. There always has to be a model for the unresolved parts of the dynamics. (Neglecting them is in general not a good idea as for nonlinear differential equations these parts will, sooner or later, spoil the numerical integration.) Then, if one aims to construct an invariant discrete counterpart of some relevant physical model, care should also be taken about the invariance characteristics of the processes that are not explicitly resolved. This is where the method of finding invariant parameterization schemes comes into play. The combination of invariant discretization schemes for the resolved part of the model with invariant parameterization schemes for the unresolved parts yields a completely invariant numerical description of a given system of differential equations. Such a fully invariant model might be closer to a true geometric numerical integration scheme than solely a symmetry preserving discretization without any closure for the subgrid-scale terms or with some non-invariant closure.

Perhaps the most relevant usage of the barotropic vorticity equation is related to two-dimensional turbulence. Turbulence on the beta-plane (or, more general, on the rotating sphere) is peculiar in that it allows for the combination of turbulent and wave-like effects. It is believed to explain the emergence of strong jets and band-like structure on giant planets in our solar system and is therefore the subject of intensive investigation, see e.g. [12–16] and references therein. In the present paper we focus on freely decaying turbulence on the beta-plane by using invariant hyperdiffusion terms to initiate the energy–enstrophy cascades. These cascades are likely responsible for the emergence of coherent, stable structures (vortices) that are ubiquitous in large-scale geophysical fluid dynamics. Note that the possibility of a development of coherent structures causing the classical inverse cascade to break down is also discussed in the literature. Energy can then be transferred between different scales without a non-linear cascade [17].

The outline of the paper is as follows. In the subsequent Section 2, we discuss and slightly extend the concept of invariant parameterization schemes as introduced in [3,18]. Special attention will be paid to methods related to invariant parameterization schemes and inverse group classification. In Section 3 we present the maximal Lie invariance algebra $\mathfrak{g}_1$ and the maximal Lie invariance pseudogroup of the barotropic vorticity equation on the beta-plane. The computation of the algebra $\mathfrak{g}_1$ is briefly described in Appendix A. A concise description of the general method for computing differential invariants of Lie (pseudo)groups using the method of moving frames is given in Section 4. In Section 5 the algebra of differential invariants is determined for the maximal Lie invariance pseudogroup of the vorticity equation. The related computation can be found in Appendix B. Two examples for invariant parameterization schemes constructed out of existing schemes using the invariantization process are presented in Section 6. Section 7 is devoted to the application of differential invariants in turbulence on the beta-plane. In particular, invariant hyperdiffusion schemes are introduced. The vorticity equation on the beta-plane is integrated numerically using both invariant and non-invariant hyperdiffusion and the corresponding enstrophy spectra are obtained. In Section 8 we discuss the possibility of deriving invariant parameterization schemes that also respect conservation laws. As an example, an invariant diffusion term is constructed that preserves the entire maximal Lie invariance pseudogroup of the vorticity equation and also preserves conservation of energy, circulation and momentum. The results are summarized and further discussed in Section 9, in which we also indicate possible future research directions in the field of invariant parameterization.

2. Invariant parameterization schemes

The problem of finding parameterization or closure schemes for subgrid-scale terms in averaged differential equations that admit Lie symmetries of the original (unaveraged) differential equation was first raised in [3], see also [19,20]. Recently, we put this idea into the framework of group classification [18], by showing that any problem of constructing invariant parameterization schemes amounts in solving a (possibly complicated) group classification problem.

As for the classical group classification, there are two principal ways to construct parameterization schemes, the direct and the inverse one [18]. In the direct approach, one replaces the terms to be parameterized with arbitrary functions depending on the mean variables and derivatives thereof. This is in the line with the general definition of all physical parameterization schemes, which are concerned to express the unknown subgrid-scale terms using the information included in the grid-scale (mean) quantities. The form of dependency of the arbitrary functions on the mean variables is guided by physical intuition. It determines the properties of all the families of invariant parameterization schemes that can be derived (e.g. the highest order of derivatives that can arise). Once the general form of the arbitrary function is chosen, one is left with a possibly rather general class of differential equations, which is amenable with tools from usual group classification, see e.g. [21–23]. This in particular will lead to a list of families of mutually inequivalent parameterization schemes that admit different Lie invariance algebras. One can then select those families that preserve the most essential symmetry features of the process to be parameterized. The final step is to suitably narrow the selected families by including other desirable physical properties into the invariant parameterization scheme.

In the present paper, however, we will be solely concerned with the inverse approach, which is why we will discuss it in a more extended manner. The inverse approach rests on the fact that any system of differential equations can be rewritten in terms of differential invariants of its maximal Lie invariance group, provided that the prolongation of the group to the corresponding jet space acts semi-regularly [24]. This property can be used in the course of the parameterization problem in the following way: Suppose that we are given a Lie group G regarded as important to be preserved for valuable parameterization schemes as a Lie symmetry group. Computing a basis of differential invariants of G along with a complete set of its independent operators of invariant differentiation, see e.g. [25–28], serves to exhaustively describe the entire algebra of differential invariants of G. As any combination of these differential invariants will necessarily be invariant with