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Conditional entropy of ordinal patterns

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HIGHLIGHTS

- Interesting quantity, the conditional entropy of ordinal patterns (CEofOP), introduced.
- Examples show that CEofOP seems to provide a good practical estimation of KS entropy.
- In particular, CEofOP has rather nice behavior for the family of logistic maps.
- Relationship of KS entropy and CEofOP is given in a rather general case.
- Equality of KS entropy and CEofOP for Markov shifts over two symbols shown.

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ABSTRACT

In this paper we investigate a quantity called conditional entropy of ordinal patterns, akin to the permutation entropy. The conditional entropy of ordinal patterns describes the average diversity of the ordinal patterns succeeding a given ordinal pattern. We observe that this quantity provides a good estimation of the Kolmogorov–Sinai entropy in many cases. In particular, the conditional entropy of ordinal patterns of a finite order coincides with the Kolmogorov–Sinai entropy for periodic dynamics and for Markov shifts over a binary alphabet. Finally, the conditional entropy of ordinal patterns is computationally simple and thus can be well applied to real-world data.

1. Introduction

The question how can one quantify the complexity of a system often arises in various fields of research. On the one hand, theoretical measures of complexity like the Kolmogorov–Sinai (KS) entropy [1,2], the Lyapunov exponent [1] and others are not easy to estimate from given data. On the other hand, empirical measures of complexity often lack a theoretical foundation; see for instance the discussion of the renormalized entropy and its relationship to the Kullback–Leibler entropy in [3–6]. Sometimes they are also not well interpretable; for example, see [7] for a criticism of the approximate entropy interpretability.

One of possible approaches to measuring complexity is based on ordinal pattern analysis [8–10]. In particular, the permutation entropy of some order d can easily be estimated from the data and has a theoretical counterpart (for order d tending to infinity), which is a justified measure of complexity. However, in this paper we consider another ordinal-based quantity, the conditional entropy of ordinal patterns. We show that for a finite order d in many cases it is closer to the KS entropy than the permutation entropy.

The idea behind ordinal pattern analysis is to consider order relations between values of time series instead of the values themselves. The original time series is converted to a sequence of ordinal patterns of an order d, each of them describing order relations between $(d+1)$ successive points of the time series, as demonstrated in Fig. 1 for order $d=3$.

The more complex the underlying dynamical system is, the more diverse the ordinal patterns occurring for the time-series are. This diversity is just what the permutation entropy measures. For example, in Fig. 1 the permutation entropy of order $d = 3$

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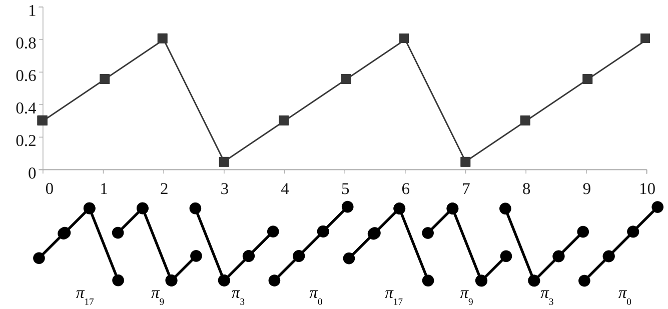

Fig. 1. Ordinal patterns of order $d = 3$ for a periodic time series, four patterns occur with period 4.

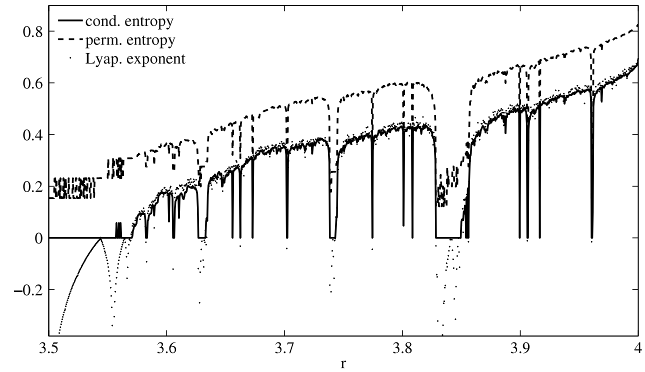

Fig. 2. Empirical conditional entropy and permutation entropy in comparison with the Lyapunov exponent for logistic maps.

is equal to $\frac{1}{3}\ln 4$, since there are four different ordinal patterns occurring with the same frequency. The permutation entropy is robust to noise [9], computationally simple and fast [10]. For order d tending to infinity the permutation entropy is connected to the central theoretical measure of complexity for dynamical systems: it is equal to the KS entropy in the important particular case [11], and it is not lower than the KS entropy in a more general case [12].

However, the permutation entropy of finite order d does not estimate the KS entropy well, while being an interesting practical measure of complexity. Even if the permutation entropy converges to the KS entropy as order d tends to infinity, the permutation entropy of finite d can be either much higher or much lower than the KS entropy (see Section 3.5 for details).

Therefore we propose to consider the conditional entropy of ordinal patterns of order d: as we demonstrate, in many cases it provides a much better practical estimation of the KS entropy than the permutation entropy, while having the same computational efficiency. The conditional entropy of ordinal patterns characterizes the average diversity of ordinal patterns succeeding a given one. For the example in Fig. 1 the conditional entropy of ordinal patterns of order $d = 3$ is equal to zero since for each ordinal pattern only one successive ordinal pattern occurs (π_9 is the only successive ordinal pattern for π_{17}, π_3 is the only successive ordinal pattern for π_9 and so on).

Let us motivate the discussion of the conditional entropy of ordinal patterns by an example.

Example 1. Consider the family of logistic maps $f_r : [0, 1] \hookleftarrow$ defined by $f_r(x) = rx(1 - x)$. For almost all $r \in [0, 4]$ the KS entropy either coincides with the Lyapunov exponent if it is positive or is equal to zero otherwise (this holds by Pesin's formula [13, Theorems 4, 6], due to the properties of f_r-invariant measures [14]). Note that the Lyapunov exponent for the logistic map can be estimated rather accurately [15]. For the logistic maps the permutation entropy of order d converges to the KS entropy as d tends to infinity. However, Fig. 2 shows that for $r \in [3.5, 4]$ the permutation entropy of order $d = 9$ is relatively far from the Lyapunov exponent in comparison with the conditional entropy of ordinal patterns of the same order (values of both entropies are numerically estimated from orbits of length $L = 4 \cdot 10^6$ of a 'random point' in [0, 1]).

In this paper we demonstrate that under certain assumptions the conditional entropy of ordinal patterns estimates the KS entropy better than the permutation entropy (Theorem 1). Besides, we prove that for some dynamical systems the conditional entropy of ordinal patterns for a finite order d coincides with the KS entropy (Theorems 5 and 6), while the permutation entropy only asymptotically approaches the KS entropy.

The paper is organized as follows. In Section 2 we fix the notation, recall the definition of the KS entropy and basic notions from ordinal pattern analysis. In Section 3 we introduce the conditional entropy of ordinal patterns and show that in some cases it approaches the KS entropy faster than the permutation entropy. Moreover, we prove that the conditional entropy of ordinal patterns for finite order d coincides with the KS entropy for Markov shifts over two symbols (Section 3.3) and for systems with periodic dynamics (Section 3.4). In Section 4 we consider the interrelation between the conditional entropy of ordinal patterns, the permutation entropy and the sorting entropy [8]. In Section 5 we observe some open question and make a conclusion. Finally, in Section 6 we provide those proofs that are mainly technical.

2. Preliminaries

2.1. Kolmogorov–Sinai entropy

In this subsection we recall the definition of the KS entropy of a dynamical system and define some related notions we will use further on. Throughout the paper we use the same notation as in [16] and refer to this paper for a brief introduction. For a general discussion and details we refer the reader to [1,17].

We focus on a measure-preserving dynamical system $(\Omega, \mathbb{B}(\Omega), \mu, T)$, where Ω is a non-empty topological space, $\mathbb{B}(\Omega)$ is the Borel sigma-algebra on it, $\mu : \mathbb{B}(\Omega) \to [0, 1]$ is a probability measure, and $T : \Omega \hookleftarrow$ is a $\mathbb{B}(\Omega)$–$\mathbb{B}(\Omega)$-measurable μ-preserving map, i.e. $\mu(T^{-1}(B)) = \mu(B)$ for all $B \in \mathbb{B}(\Omega)$.

The complexity of a system can be measured by considering a coarse-grained description of it provided by symbolic dynamics. Given a finite partition $\mathcal{P} = \{P_0, P_1, \ldots, P_l\} \subset \mathbb{B}(\Omega)$ of Ω (below we consider only partitions $\mathcal{P} \subset \mathbb{B}(\Omega)$ without mentioning this explicitly), one assigns to each set $P_a \in \mathcal{P}$ the symbol a from the alphabet $A = \{0, 1, \ldots, l\}$. Similarly, the n-letter word $a_0 a_1 \cdots a_{n-1}$ is associated with the set $P_{a_0 a_1 \cdots a_{n-1}}$ defined by

$$P_{a_0 a_1 \cdots a_{n-1}} = P_{a_0} \cap T^{-1}(P_{a_1}) \cap \cdots \cap T^{-\circ(n-1)}(P_{a_{n-1}}). \tag{1}$$

Then the collection

$$\mathcal{P}_n = \{P_{a_0 a_1 \ldots a_{n-1}} \mid a_0, a_1, \ldots, a_{n-1} \in A\} \tag{2}$$

forms a partition of Ω as well. The *Shannon entropy*, the *entropy rate* and the *Kolmogorov–Sinai (KS) entropy* are respectively defined by (we use the convention that $0 \ln 0 := 0$)

$$H(\mathcal{P}) = -\sum_{P \in \mathcal{P}} \mu(P) \ln \mu(P),$$

$$h_\mu(T, \mathcal{P}) = \lim_{n \to \infty} (H(\mathcal{P}_{n+1}) - H(\mathcal{P}_n)) = \lim_{n \to \infty} \frac{H(\mathcal{P}_n)}{n},$$

$$h_\mu(T) = \sup_{\mathcal{P} \text{ finite partition}} h_\mu(T, \mathcal{P}).$$