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Singular continuation of planar central configurations with clusters of bodies

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HIGHLIGHTS

- Central configurations are created by continuation.
- Initial mass in the continuation may be zero or nonzero.
- The resulting configurations have clusters of point masses.
- Two scaling regimes are explored, and existence results proved.
- This relates to the finiteness problem for central configurations.

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ABSTRACT

Planar central configurations of point masses that have one or more clusters of bodies are created by analytic continuation. The singularity of the gravitational interaction is removed from the continuation equations by algebraic means. The continuation splits a single body into several, and the initial mass of the single body can be nonzero. Necessary conditions are derived for these continuations that may be useful in addressing the question of finiteness of central configurations.

1. Introduction

Central configurations are arrangements of point masses in which the acceleration of each body due to gravitational interactions is proportional to the body's displacement from the center of mass. These configurations play a significant role in several areas of celestial mechanics, such as collisions and periodic solutions [1,2]. They generally do not contain clusters of point masses; a quantitative analysis of the minimum size of clusters appears in [3]. Two notable exceptions have appeared in the literature. First, central configurations with some clusters of very small masses embedded within a larger central configuration may be constructed by analytic continuation [4,5] or by spacing them closely along curves [6]. The second exception [7] is a continuous family of planar central configurations of five masses, in which three masses form a cluster for some parameter values. Analogous families have been constructed in ambient spaces of dimension four and higher [8].

In an effort to better understand clustered configurations, this paper addresses the technique of the creation of central configurations with clusters by continuation *without* the restriction to small masses. The singularity of the gravitational interaction as the cluster shrinks to a point poses an obstacle that is overcome by algebraic manipulations taking advantage of some basic symmetries of the interaction. The masses may be fixed, or may be allowed to vary with the continuation parameter. Sufficient conditions for the existence of such configurations are found.

The case of clusters with small masses is considered in Theorem 1, duplicating some results obtained by Xia and Moeckel. Xia [5] used clusters of two small masses as a tool to enumerate the number of central configurations for certain mass distributions. Moeckel [4] proved the existence via continuations of central configurations with clusters of arbitrary numbers of small masses and established criteria for their stability. The technique used in the present work is slightly different.

Theorems 2 and 3 deal with the construction by continuation of clusters of bodies with masses not tending to zero at the singularity. As is the case in the family described by Roberts [7], these masses cannot all be positive, since the clustering bodies

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asymptotically form acceleration-free configurations. If the masses do not vary with the continuation parameter, additional necessary conditions relate the clustered and nonclustered bodies; this is the content of Theorem 2. If the clustering masses are made to vary as functions of the continuation parameter, Theorem 3 gives sufficient conditions for constructing configurations where the size of the cluster is the continuation parameter. The analysis deals with a single cluster, but is easily generalized to more than one cluster in the same manner as [4].

While the central configurations studied in this paper are nonphysical (due to the negative masses) they are certainly interesting in their possible relevance to an important open question in celestial mechanics. The results of Theorems 2 and 3 may shed light on the existence of other configuration families directly comparable to the one found by Roberts. The existence of such families is relevant to the question of finiteness of planar relative equilibria in the n-body problem, as discussed in Section 4. If it can be proved that the necessary conditions of Theorem 2 cannot be met by a given collection of masses, then the alternative to finiteness will be strongly constrained. This finiteness question is open for n greater than 5 [9,10].

2. Planar central configurations in cluster variables

Point masses of size m_i at positions z_i in the complex plane, $1 \le i \le n$, form a *central configuration* with center of mass c if

$$0 = f_i := (z_i - c) - \omega \sum_{j \neq i} m_j \frac{z_i - z_j}{|z_i - z_j|^3}, \quad 1 \le i \le n \tag{1}$$

for some positive value of the parameter ω. It is no loss of generality to put $c = 0$ and $\omega = 1$ by translation and scaling of the position variables. Define the moment of inertia I and potential U as follows:

$$I = \sum m_i |z_i|^2, \qquad U = \sum_{i<j} \frac{m_i m_j}{|z_i - z_j|}. \tag{2}$$

Observe that while Eq. (1) are complex, the sum

$$\sum m_i \bar{z}_i f_i = I - U \tag{3}$$

is a real quantity. Any central configuration may be rotated around the center of mass, $z_i \mapsto \exp(i\theta) z_i$, to form another central configuration. Solutions of the system of equation (1) will thus be degenerate in the sense that they are not isolated in configuration space. The collection of configurations generated by rotation forms an equivalence class, and a representative of this class can be found by picking the index m of a mass not at the origin and rotating the central configuration until the position z_m satisfies some additional constraint, such as lying on a specified curve. Thus by replacing the system (1) with an equivalent system

$$0 = f_i, \quad 1 \le i < n, \tag{4}$$

$$0 = I - U + i\,\mathrm{Im}(z_m) \tag{5}$$

one can remove the rotation degeneracy of the system (1) by seeking central configurations that have the mth mass lying on the real axis. Central configurations that satisfy (4) and (5) will be called regular if the Jacobian matrix of the right-hand sides of (4) and (5) is nonsingular.

Consider a configuration in which the first s bodies are in close proximity. It will prove useful to replace the position variables $z_1, \dots, z_s$ by "cluster variables" that reflect the scale of the cluster of bodies, and to reformulate Eqs. (4) and (5) using these variables — eventually yielding the equivalent system of equations (9), (16) and (17).

To begin, write $A = \{1, \dots, s\}$ and $B = \{s+1, \dots, n\}$ for the collection of indices in and not in the cluster; write $m_0 = \sum_{i\in A} m_i$ for total mass in the cluster and $z_0 = \sum_{i\in A} (m_i/m_0) z_i$ for its center of mass. Now define ζ_i through the relations $z_i = z_0 + (re^{i\theta})\zeta_i$ for each i in A, where real variables r and θ are chosen so as to make $\zeta_1 = 1$. The new variable r shows the size of the cluster and θ controls its overall orientation. Since $\sum_A m_i \zeta_i = 0$, the variables $r, \theta, z_0, \zeta_2, \dots, \zeta_{s-1}$ can be used in place of the variables $z_1, \dots, z_s$.

Now we express the functions f_i in terms of these cluster variables, under the assumption that r is small. First consider $i \in B$:

$$f_i = z_i - \sum_{j\in A} m_j \frac{z_i - z_0 - re^{i\theta}\zeta_j}{|z_i - z_0 - re^{i\theta}\zeta_j|^3} - \sum_{j\in B, j\neq i} m_j \frac{z_i - z_j}{|z_i - z_j|^3}. \tag{6}$$

In the first sum, since r is small we may write

$$\frac{z_i - z_0 - re^{i\theta}\zeta_j}{|z_i - z_0 - re^{i\theta}\zeta_j|^3} = \frac{z_i - z_0}{|z_i - z_0|^3} - L(re^{i\theta}\zeta_j) + O(r^2), \tag{7}$$

where L is some linear operator and $O(r^2)$ denotes the product $r^2 g$ for some continuous function g. Because of the identity $\sum_A m_j \zeta_j = 0$, the linear terms in (6) cancel and we have

$$f_i = z_i - m_0 \frac{z_i - z_0}{|z_i - z_0|^3} - \sum_{j\in B, j\neq i} m_j \frac{z_i - z_j}{|z_i - z_j|^3} + O(r^2). \tag{8}$$

It is convenient to use the index set $B' = B \cup \{0\}$ and write (8) in the form

$$f_i = z_i - \sum_{j\in B'} m_j \frac{z_i - z_j}{|z_i - z_j|^3} + O(r^2). \tag{9}$$

This expression is valid for all i in B.

Next suppose $i \in A$. Using the new variables, we have

$$f_i = z_0 + re^{i\theta}\zeta_i - \frac{re^{i\theta}}{|r|^3} \sum_{j\in A, j\neq i} m_j \frac{\zeta_i - \zeta_j}{|\zeta_i - \zeta_j|^3}$$
$$- \sum_{j\in B} m_j \frac{z_0 + re^{i\theta}\zeta_i - z_j}{|z_0 + re^{i\theta}\zeta_i - z_j|^3}. \tag{10}$$

The sum over B may be simplified as in (7), and by virtue of the identity

$$\frac{z+\zeta}{|z+\zeta|^3} = \frac{z}{|z|^3} + \left(\frac{-1}{2|z|^3}\right)\zeta + \left(\frac{-3z^2}{2|z|^5}\right)\bar{\zeta} + O(|\zeta|^2) \tag{11}$$

this sum in (10) takes the form

$$\sum_{j\in B} m_j \frac{z_0 - z_j}{|z_0 - z_j|^3} + \alpha re^{i\theta}\zeta_i + \beta re^{-i\theta}\bar{\zeta}_i + O(r^2) \tag{12}$$

where the constants α, β are

$$\alpha = -\frac{1}{2}\sum_B m_j \frac{1}{|z_0 - z_j|^3}, \qquad \beta = -\frac{3}{2}\sum_B m_j \frac{(z_0 - z_j)^2}{|z_0 - z_j|^5}. \tag{13}$$

Using the convenient abbreviation

$$f_0 = z_0 - \sum_{j\in B} m_j \frac{z_0 - z_j}{|z_0 - z_j|^3}.$$

Eq. (10) reads

$$f_i = f_0 + O(r^2)$$
$$+ re^{i\theta}\left(\zeta_i - \alpha\zeta_i - e^{-2i\theta}\beta\bar{\zeta}_i - \frac{1}{|r|^3}\sum_{j\in A} m_j \frac{\zeta_i - \zeta_j}{|\zeta_i - \zeta_j|^3}\right). \tag{14}$$

When inserting (14) into the sum $\sum_A m_i f_i$ the terms in parentheses vanish since by symmetry:

$$\sum_A m_i \left(\sum_{j\in A} m_j \frac{\zeta_i - \zeta_j}{|\zeta_i - \zeta_j|^3}\right) = 0.$$