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Nonlocal interactions by repulsive–attractive potentials: Radial ins/stability

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ABSTRACT

We investigate nonlocal interaction equations with repulsive–attractive radial potentials. Such equations describe the evolution of a continuum density of particles in which they repulse (resp. attract) each other in the short (resp. long) range. We prove that under some conditions on the potential, radially symmetric solutions converge exponentially fast in some transport distance toward a spherical shell stationary state. Otherwise we prove that it is not possible for a radially symmetric solution to converge weakly toward the spherical shell stationary state. We also investigate under which condition it is possible for a non-radially symmetric solution to converge toward a singular stationary state supported on a general hypersurface. Finally we provide a detailed analysis of the specific case of the repulsive–attractive power law potential as well as numerical results.

1. Introduction

Nonlocal interaction equations are continuum models for large systems of particles where every single particle can interact not only with its immediate neighbors but also with particles far away. These equations have a wide range of applications. In biology they are used to model the collective behavior of a large number of individuals, such as a swarm of insects, a flock of birds, a school of fish or a colony of bacteria [1–17]. In these models individuals sense each other at a distance, either directly by sound, sight or smell, or indirectly via chemicals, vibrations, or other signals. Nonlocal interaction equations also arise in various contexts in physics. They are used in models describing the evolution of vortex densities in superconductors [18–26]. They also appear in the modeling of dynamics of agglomerating particles in two dimensions (with loose links to the one-dimensional sticky particle system) [27]. They also appear in simplified inelastic interaction models for granular media [28–31]. Going back to biology, nonlocal interaction equations arise also in the modeling of the orientational distribution of F-actin filaments in cells [32–34].

In their simplest form, nonlocal interaction equations can be written as

$$\frac{\partial \mu}{\partial t} + \mathrm{div}(\mu v) = 0, \quad v = -\nabla W * \mu \tag{1}$$

where $\mu(t) = \mu_t$ is the probability measure of particles at time t, $W : \mathbb{R}^N \longrightarrow \mathbb{R}$ is the interaction potential and $v(t, x)$ is the velocity of the particles at time t and at location $x \in \mathbb{R}^N$. We will always assume that the interaction potential $W(x) = k(|x|)$ is radial and C^2-or C^3-smooth away from the origin, depending on the results. Typically the potentials we will consider have a singularity (not C^2-smooth) at the origin.

When the potential W is purely attractive, i.e. W is a radially symmetric increasing function, then the density of particles collapse on itself and converge to a Dirac Delta function located at the center of mass of the density. This Dirac Delta function is the unique stable steady state and it is a global attractor [35]. The collapse toward the Dirac Delta function can take place in finite time if the interaction potential is singular enough at the origin and several works have been recently devoted to the understanding of these singular measure solutions [36,37,35,38].

In biological applications however, it is often the case that individuals attract each other in the long range in order to remain in a cohesive group, but repulse each other in the short range in order to avoid collision [39,40]. This lead to the choice of a radially symmetric potential W which is first decreasing then increasing as a function of the radius. We refer to these type of potentials as repulsive–attractive potentials. Compared with the purely attracting case where solutions always converge to a single Delta function, nonlocal interaction equations with repulsive–attractive potentials lead to solutions converging to possibly complex steady states. As such, nonlocal interaction equations with repulsive–attractive potentials can be considered as a minimal model for pattern formation in large groups of individuals. They also arise in material

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sciences [41–45] where particles, nano-particles or molecules self-assemble according to pairwise interactions generated by a repulsive–attractive potential.

Whereas nonlocal interaction equations with purely attractive potential have been intensively studied there are still relatively few rigorous results about nonlocal interaction equations with repulsive–attractive potential. The 1D case has been studied in a series of works [46–48]. The authors have shown that the behavior of the solution depends highly on the regularity of the interaction potential: for regular interaction, the solution converges to a sum of Dirac masses, whereas for singular repulsive potential, the solution remains uniformly bounded. They also showed that combining a singular repulsive with a smooth attractive potential leads to integrable stationary states. Pattern formation in multi-dimensions have recently been studied in [49,50]. In these two works, the authors perform a numerical study of the finite particle version of (1) and show that a repulsive–attractive potential can lead to the emergence of surprisingly complex patterns. To study these patterns they plug in (1) an ansatz which is a distribution supported on a surface. This give rise to an evolution equation for the surface. They then perform a linear stability analysis around the uniform distribution on the sphere and derive simple conditions on the potential which classify the different instabilities. The various instability modes dictate toward which pattern the solution will converge. They also check numerically that what is true for the surface evolution equation also holds for the continuum model (1). In other recent works [51,52] the specific case where the repulsive part of the potential is the Newtonian potential is analyzed showing the existence of radially compactly supported integrable stationary states. They also study their nonlinear stability for radial solutions.

In this paper we focus primarily on proving rigorous results about the convergence of radially symmetric solutions toward spherical shell stationary states in multi-dimensions.

Definition 1 (*Spherical Shell*)**.** The spherical shell of radius R, denoted δ_R, is the probability measure which is uniformly distributed on the sphere $\partial B(0, R) = \{x \in \mathbb{R}^N : |x| = R\}$.

Given a repulsive–attractive radial potential whose attractive force does not decay too fast at infinity, there always exists an $R > 0$ so that the spherical shell of radius R is a stationary state as it will be remarked below. One needs then to address the question of whether or not this spherical shell is stable. It is classical, see [53,29,54,55], that the Eq. (1) is a gradient flow of the interaction energy

$$E[\mu] = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} W(x-y)\,d\mu(x)\,d\mu(y) \tag{2}$$

with respect to the euclidean Wasserstein distance. Thus, stable steady states of (1) are expected to be local minimizers of the interaction energy. Simple energetic arguments will show that in order for the spherical shell of radius R to be a local minimum of the interaction energy, it is necessary that the radial potential W satisfies:

(C0) Repulsive–Attractive Balance: $\omega(R, R) = 0$,
(C1) Fattening Stability: $\partial_1\omega(R, R) \le 0$,
(C2) Shifting Stability: $\partial_1\omega(R, R) + \partial_2\omega(R, R) \le 0$,

where the function $\omega : \mathbb{R}_+^2 \longrightarrow \mathbb{R}$ is defined by

$$\omega(r, \eta) = -\frac{1}{\sigma_N}\int_{\partial B(0,1)} \nabla W(re_1 - \eta y)\cdot e_1\, d\sigma(y), \tag{3}$$

σ_N is the hypersurface area of the unit sphere in $\mathbb{R}^N$, e_1 is the first vector of the canonical basis of $\mathbb{R}^N$, $d\sigma$ denotes the volume element of the manifold where the integral is performed and $\mathbb{R}_+^2 = (0, +\infty) \times (0, +\infty)$. Condition (C0) simply guarantees that the spherical shell δ_R is a critical point of the interaction energy. We will see that if condition (C1) is not satisfied then it is energetically favorable to split the spherical shell into two spherical shells. Heuristically this indicate that the density of particles, rather than remaining on the sphere, is going to expand and occupy a domain in $\mathbb{R}^N$ of positive Lebesgue measure. If condition (C1) is not satisfied we will therefore say that the "fattening instability" holds. It can be easily checked that if $\omega(R, R) = 0$, then $\partial_1\omega(R, R)$ is simply the value of the divergence of the velocity field on the sphere of radius R. So the fattening instability corresponds to an expanding velocity field on the support of the steady state. We will also see that if condition (C2) is not satisfied it is energetically favorable to increase or decrease the radius of the spherical shell. This instability will be referred as the "shift instability".

We now outline the structure of the paper and describe the main results. In the preliminary section, Section 2, we derive (C0)–(C2) from an energetic point of view and we show that they correspond to avoiding the fattening and shift instability. We also study the regularity of the kernel ω defined by (3). A good understanding of the regularity of ω will be necessary for later sections. Some technical results of this regularity analysis need nontrivial aspects of differential geometry and are relegated to an Appendix for readiness. We also remind the reader of previous results from [56,57] about well posedness of (1) in $L^p(\mathbb{R}^N)$ and set up the overall notation.

Section 3 is devoted to a detailed study of the fattening instability, both in the radially symmetric case and in the non-radially symmetric case. We first show that if condition (C1) is not satisfied then it is not possible for a radially symmetric L^p-solution to converge weakly-$*$ as measures toward a spherical shell stationary state. We then investigate singular stationary states supported on hypersurfaces which are not necessarily spheres. Such steady states have been observed in numerical simulations [49,50]. We show that if the divergence of the velocity field generated by such stationary state is positive everywhere on their support, then it is not possible for an L^p-solution to converge toward the stationary state in the sense of the topology defined by d_∞. Here d_∞ stands for the infinity-Wasserstein distance on the space of probability measures (see next section for a definition). We also show that if the repulsive–attractive potential W is singular enough at the origin, for example $W(x) \sim -|x|^b/b$ as $|x| \to 0$ with $b \le 3 - N$, then the potential is so repulsive in the short range that solutions cannot concentrate on an hypersurface, and this is independent of how attractive is the potential in the long range. To be more precise we show that for potentials with such a strong repulsive singularity at the origin, L^p solutions cannot converge with respect to the d_∞-topology toward singular steady states supported on hypersurfaces.

Whereas Section 3 is devoted to instability results, Section 4 is devoted to stability results. We show that if (C0)–(C2) hold with strict inequalities, then a radially symmetric solution of (1) which starts close enough to the spherical shell in the d_∞ topology will converge exponentially fast toward it. Under additional assumptions on the potential we can also prove convergence with respect to the d_α topology, $\alpha \in [1, +\infty)$. In order for the stability results of Section 4 to hold a certain amount of regularity on the solutions is necessary. Unfortunately weak L^p-solutions do not have this amount of regularity. This is why in Section 5 we prove well posedness of classical C^1-solutions. This covers a gap in the existing literature which mostly considers weak solutions. The results of Section 4 are true for this class of classical C^1-solutions. The aim of Section 6 is to show examples of how to apply the general instability and stability theory in the case of power-law repulsive–attractive potentials:

$$W(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b} \quad 2 - N < b < a. \tag{4}$$

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