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Equilibria of biological aggregations with nonlocal repulsive–attractive interactions

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ABSTRACT

We consider the aggregation equation $\rho_t - \nabla \cdot (\rho \nabla K * \rho) = 0$ in $\mathbb{R}^n$, where the interaction potential K incorporates short-range Newtonian repulsion and long-range power-law attraction. We study the global well-posedness of solutions and investigate analytically and numerically the equilibrium solutions. We show that there exist unique equilibria supported on a ball of $\mathbb{R}^n$. By using the method of moving planes we prove that such equilibria are radially symmetric and monotone in the radial coordinate. We perform asymptotic studies for the limiting cases when the exponent of the power-law attraction approaches infinity and a Newtonian singularity, respectively. Numerical simulations suggest that equilibria studied here are global attractors for the dynamics of the aggregation model.

1. Introduction

The multidimensional integro-differential equation,

$$\rho_t - \nabla \cdot (\rho \nabla K * \rho) = 0, \tag{1}$$

has attracted a great amount of interest in recent years. The equation appears in various contexts related to mathematical models for biological aggregations, where ρ represents the density of the aggregation and K is the social interaction potential. The asterisk $*$ denotes convolution. We refer to [1,2] for an extensive background and literature review on mathematical models of social aggregations and in particular, for a thorough discussion on the relevance of Eq. (1) for modelling swarming behaviours. The equation also arises in a number of other applications such as granular media [3,4], self-assembly of nanoparticles [5,6], Ginzburg–Landau vortices [7–9] and molecular dynamics simulations of matter [10]. In this work however we are primarily interested in biological applications, where Eq. (1) is used to model social aggregations such as insect swarms, fish schools, bacterial colonies, etc. [1].

Regarded as a model for biological aggregations, Eq. (1) incorporates inter-individual social interactions such as long-range attraction and short-range repulsion, through the aggregation potential K. The properties of the potential (symmetry, regularity, monotonicity, etc.) are essential in studying issues such as the well-posedness [11–13] or the long-time behaviour [14,15] of solutions to model equation (1). In particular, a large component of the research on this model dealt with attractive potentials K which lead to solutions that blow-up (in finite or infinite time) by mass concentration, into one or several Dirac distributions [16–18].

It is essential however for an aggregation model to be able to capture solutions with biologically relevant features. As pointed out by Mogilner and Keshet in their seminal work [1] on the class of models discussed here, such desired characteristics include: finite densities, sharp boundaries, relatively constant internal population and long lifetimes. The difficulty in finding such solutions to model (1) has been indicated as a “challenge” in previous literature [19,20], and in fact there is only a handful of works that address this issue. Topaz and collaborators [15,21] derived explicit swarm equilibria that arise in the one-dimensional model with Morse-type potentials (in the form of decaying exponentials), but their explicit calculations do not extend to higher dimensions. Other works illustrate asymptotic vortex states in 2*D* [19] and clumps (aggregations with compact support) in a nonlocal model that includes density-dependent diffusion [2].

A recent publication of the authors [22] considered an interaction potential K for which equilibria of the aggregation model (1) have the desired characteristics indicated above. More specifically, the kernel investigated in [22] has a repulsion component in the form of the Newtonian potential and attraction given by a power law[1]:

$$K(x) = \phi(x) + \frac{1}{q}|x|^q. \tag{2}$$

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[1] See Section 5 for a discussion on how the potential can be modified to avoid the biologically unrealistic growth of attraction with distance when $q > 0$.

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Here, $\phi(x)$ is the free-space Green's function of the negative Laplace operator $-\Delta$:

$$\phi(x) = \begin{cases} -\frac{1}{2}|x|, & n = 1 \\ -\frac{1}{2\pi}\ln|x|, & n = 2 \\ \frac{1}{n(n-2)\omega_n}\frac{1}{|x|^{n-2}}, & n \geq 3, \end{cases} \tag{3}$$

and q is a real exponent, $q \geq 2$. In (3), n is the number of space dimensions and ω_n denotes the volume of the unit ball in $\mathbb{R}^n$.

We summarize briefly some of the results from [22] that are relevant to the present article. For $q = 2$, the equilibrium density of (1)–(2) is uniform inside a ball of $\mathbb{R}^n$ and zero outside it. In this case, the method of characteristics was used to solve explicitly the dynamics corresponding to radially symmetric initial conditions in any dimension. This showed the global stability within the class of radially symmetric solutions of the constant steady state. The explicit calculations did not extend to general exponent $q > 2$, but the existence of a unique radially symmetric equilibrium of compact support was shown, after casting the equilibrium problem as an eigenvalue problem for an integral operator and applying the Krein–Rutman theorem. Some explicit calculations of the equilibria could be performed however for the special subcase when q is even. In addition to these studies on equilibria, the global well-posedness of solutions to (1)–(2) (with $q \geq 2$) was shown by borrowing techniques used in the analysis of incompressible Euler equations [23].

The main purpose of the work from [22] was to design attractive–repulsive potentials that yield equilibrium states of finite densities and compact support. In this respect, the attraction component $\frac{1}{q}|x|^q$ of the potential was specifically designed to counter-balance the singular Newtonian repulsion. Newtonian (attractive) potentials for model (1) were considered in [8,9] in the context of vortex motions in two-dimensional superfluids. The main concern of these works was the well-posedness of solutions, in particular concentration and singularity formation in measure-valued solutions. Very recently, Newtonian potentials were also considered in aggregation models [24,25]. In [25], the authors study patch solutions and they consider separately the case of an attractive Newtonian potential (with finite time concentration) and of a repulsive Newtonian potential (with spreading to a circular/ spherical aggregation patch).

The purpose of the present research is (i) to extend the interaction potential (2)–(3) studied in [22] to allow for more general attractive forces[2] ($q > 2 - n$) and (ii) to investigate analytically and numerically the properties of the equilibria to the aggregation model (1)–(2) for $q > 2 - n$. Remarkably, the intricate balance between the power-law attraction and the singular repulsion provides the model with a very interesting and at the same time biologically relevant set of steady states. For all values of $q \in (2 - n, \infty)$, the aggregation model has a unique steady state supported in a ball. This steady state is radial and monotone in the radial coordinate. More specifically, the equilibria are decreasing about the origin for $2 - n < q < 2$ and increasing for $q > 2$, while $q = 2$ corresponds to a constant equilibrium density. Fig. 1 shows the equilibrium solutions in three dimensions for various values of q; all shown equilibria have mass 1. The limits $q \to \infty$ and $q \searrow 2 - n$, that is, when attraction becomes infinitely strong (at large distances) or as singular as the (Newtonian) repulsion, are particularly interesting. As $q \to \infty$, the radii of the equilibria approach a constant, but the qualitative features change dramatically, as mass aggregates towards the edge of the swarm, leaving an increasingly void region in the centre — this effect can be observed in Fig. 1 ($q = 20, 40, 80$). As $q \searrow 2 - n$, the radii of equilibria approach 0 and mass concentrates at the origin — see Fig. 1 ($q = 1.5, 1, 0.5$). Numerical simulations suggest that all these equilibria are global attractors for the dynamics of (1)–(2), which motivates and gives strong grounds to the studies of the present work.

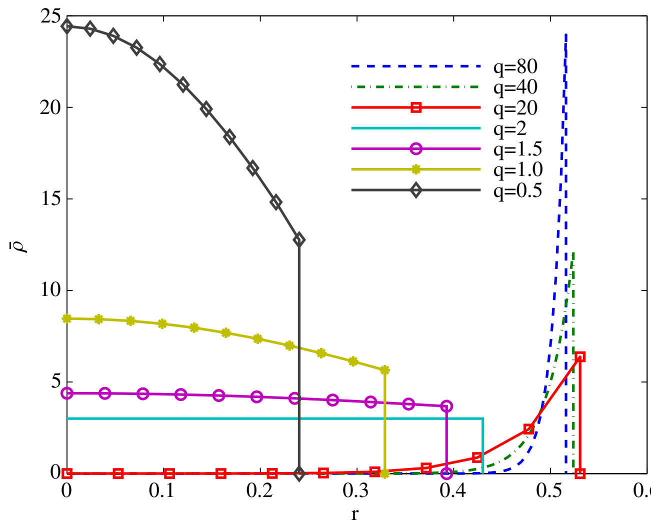

Fig. 1. Radially symmetric equilibria of (1)–(2) in three dimensions, for various values of q. The equilibria are monotone in the radial coordinate: decreasing about the origin for $2 - n < q < 2$, increasing for $q > 2$, and constant for $q = 2$. As $q \to \infty$, the radii of the equilibria approach a constant, and mass aggregates towards the edge of the swarm, leaving an increasingly void region in the centre. As $q \searrow 2 - n$, the radii of equilibria approach 0 and mass concentrates at the origin. Numerics suggests that all these equilibria are global attractors for the dynamics of (1)–(2).

There are a few very recent studies of equilibria of (1) with attractive–repulsive potentials in power-law form that closely relate to our work. In [26], the authors study the stability of spherical shell equilibria for potentials in the form $K(x) = \frac{1}{q}|x|^q - \frac{1}{p}|x|^p$, where $2 - n < p < q$ (short range repulsion and long range attraction). While the attraction component, $|x|^q/q$, is the same as in (2), the singularity of the repulsion term is "better" than Newtonian ($p > 2 - n$). Shell steady states are shown to exist and be locally stable under certain conditions on the exponents p and q. The methods from [26] do not apply to Newtonian singularities. Other recent works that involve model (1) with power-law potentials focus on pattern formation and linear stability analysis of spherical shells [27,28].

The results of the paper are as follows. Well-posedness of solutions to the aggregation model (1)–(2) (with $q > 2 - n$) is studied in Section 2 by using analogies with the incompressible fluid flow equations [23,22,25]. For all values of $q \in (2 - n, \infty)$, we show in Section 3.1 that there exist unique equilibria supported on a ball of $\mathbb{R}^n$. In Section 3.2 we employ the *method of moving planes* [29] to prove that such equilibria are radially symmetric and monotone in the radial coordinate. In Section 4 we performed careful asymptotic and numerical investigations of the equilibria. Our studies address two issues. The first one is the behaviour of equilibria as $q \to \infty$ and $q \to 2 - n$. As expected, the two limiting cases give very different asymptotic behaviours. The second issue addressed in Section 4 is the stability of the equilibrium solutions. The results regarding stability are preliminary and entirely based on numerical observations. Based on all numerical experiments we performed, we conjecture that the equilibria studied in this paper are global attractors for solutions to (1)–(2).

[2] In the special case $q = 0$ we take $K(x) = \phi(x) + \ln|x|$.