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Singular patterns for an aggregation model with a confining potential

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ABSTRACT

We consider the aggregation equation with an attractive–repulsive force law. Recent studies (Kolokolnikov et al. (2011) [22]; von Brecht et al. (2012) [23]; Balague et al. (2013) [15]) have demonstrated that this system exhibits a very rich solution structure, including steady states consisting of rings, spots, annuli, N-fold symmetries, soccer-ball patterns etc. We show that many of these patterns can be understood as singular perturbations off lower-dimensional equilibrium states. For example, an annulus is a bifurcation from a ring; soccer-ball patterns bifurcate off solutions that consist of delta-point concentrations. We apply asymptotic methods to classify the form and stability of many of these patterns. To characterize spot solutions, a class of "semi-linear" aggregation problems is derived, where the repulsion is described by a nonlinear term and the attraction is linear but non-symmetric. For a special class of perturbations that consists of a Newtonian repulsion, the spot shape is shown to be an ellipse whose precise dimensions are determined via a complex variable method. For annular shapes, their width and radial density profile are described using perturbation techniques.

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1. Introduction

Collective group behaviour is a fascinating natural phenomenon that is observed at all levels of the animal kingdom, from beautiful bacterial colonies, insect swarms, fish schools and flocks of birds, to complex human population patterns. The emergence of very complex behaviour is often a consequence of individuals following very simple rules, without any external coordination. In recent years, many models of group behaviour have been proposed that involve nonlocal interactions between the species [1–6]. Related models also arise in other important applications such as self-assembly of nanoparticles [7,8], theory of granular gases [9], invasion models [10], chemotaxic motion [11,12], and molecular dynamics simulations of matter [13].

One of the simplest models of insect swarming was proposed in [5]. In this model, each individual is represented by a particle moving in space. Every particle A "feels" every other particle B through a force whose magnitude $F(r)$ depends only on the pairwise distance between the two particles and which acts in the direction from A to B. Each particle then moves in the direction of the average force acting upon it. These simple assumptions lead to an *aggregation model* for a system of particles located at $\{x_1, \ldots, x_N\}$,

$$\frac{d}{dt}x_k = \frac{1}{N}\sum_{j\neq k} F\left(\left|x_k - x_j\right|\right)\frac{x_k - x_j}{\left|x_k - x_j\right|}, \quad k = 1, \ldots, N. \tag{1}$$

The force law $F(r)$ is assumed to be *repulsive at short distances* (i.e. $F(r) > 0$ for small r) and *attractive at large distances* (i.e. $F(r) < 0$ for sufficiently large r). For convenience, we will often use the notation

$$\frac{d}{dt}x_k = \frac{1}{N}\sum_{j\neq k} f\left(\left|x_k - x_j\right|\right)\left(x_k - x_j\right), \quad k = 1, \ldots, N, \tag{2}$$

where $f(r) = F(r)/r$. The continuum limit $N \to \infty$ of (2) yields the system [14],

$$\rho_t + \nabla\cdot(v\rho) = 0; \quad v = \int_{\mathbb{R}^d} f\left(|x-y|\right)(x-y)dy. \tag{3}$$

The aggregation model (3) and its discrete analogue (2) have been intensively studied over the past decade and by now a vast literature exists on this topic; see for example [15,14,16–22,5,6,23,24] and references therein. There are also many studies of related second-order models that incorporate acceleration of self-propelled particles; see for instance [25–27] and references therein.

In a series of papers [22–24,15,28], the authors have investigated a very rich solution structure for a family of such attrac-

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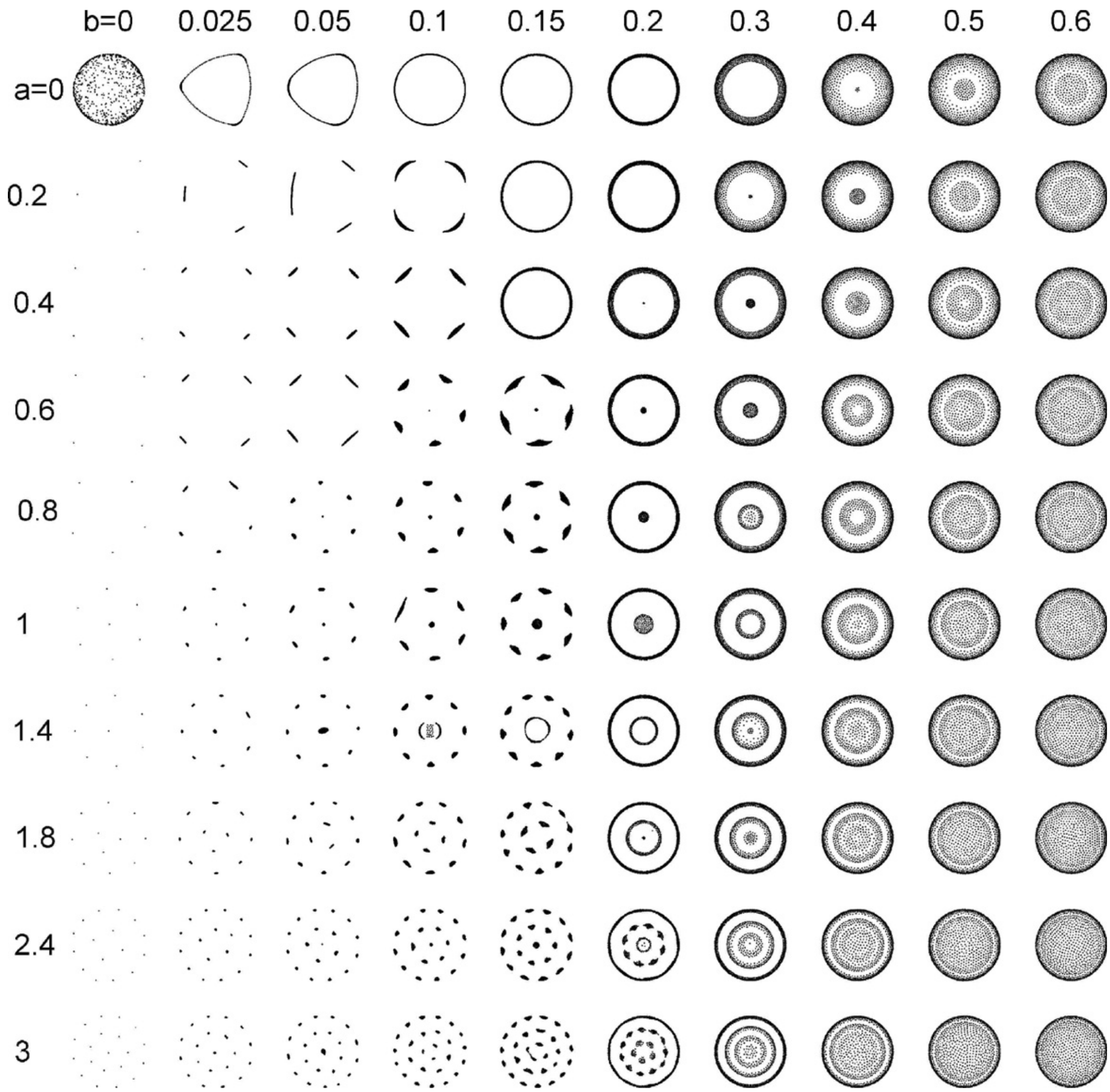

Fig. 1. Steady states of (1) with $F(r) = \min(ar + b, 1 - r)$, using $N = 1000$ particles and with a, b as indicated. A snapshot at $t = 10,000$ is shown. Integration was performed using the forward Euler method with stepsize 0.5.

tive–repulsive force laws. A particularly simple solution in two dimensions consists of a *ring*, where the particles align themselves along a circular ring uniformly. Another type of a simple solution consists of *clusters* of particles, whereby the equilibrium state consists of K "holes", with each particle belonging to one of such holes. The stability of cluster solutions in one dimension was characterized in [20]; we will extend this analysis to higher dimensions in Section 2 below.

To illustrate the large variety of possible steady states, consider the "piecewise-linear" force

$$F(r) = \min(ar + b, 1 - r), \quad 0 \le b \le 1. \tag{4}$$

As shown on Fig. 1, this family generates a rich equilibrium structure that is very sensitive to the choice of parameters a and b. Note in particular the presence of "spot" solutions, such as when $(a, b) = (0.8, 0.05)$; and the annulus solutions such as when $(a, b) = (0.4, 0.15)$. Such solutions are prevalent in numerical simulations, and typically bifurcate from simpler ring or cluster solutions.[1] The main goal of this paper is to study these more complex solutions including annuli, spots, and "soccer balls".

Let us now summarize our findings. We start by extending the work of [20] on point clusters to two and higher dimensions in Section 2. Such clusters can occur when the repulsion near the origin is weak, $F(0) = 0$. On the other hand, when the repulsion at small distances is small but positive, $0 < F(0) \ll 1$, the holes "degenerate" into small spots. In Section 3 we derive the reduced canonical problem (30) that describes the shape of a single spot. This reduced problem depends only on two parameters and is analysed in Section 3. There are two basic steady states of the reduced problem (30): the simplest steady state consists of particles along a line; such steady states appear for in Fig. 1 with $(a, b) = (0.2, 0.025)$. A more complex shape is a fully-two dimensional steady state such as e.g. $(a, b) = (0.8, 0.05)$. We fully characterize the stability of the former in terms of Harmonic numbers (see Theorem 3.2). Using the results of [19] also shows that the steady states are bounded in the continuum limit. In Section 3.2 we extend the analysis for the case where a small amount of Newtonian repulsion is added to the kernel. The resulting spots have a constant density and their shape is an ellipse whose axes are completely characterized in terms of the original kernel (see Theorem 3.3 for details).

In Section 4 we turn our attention annular solutions such as in Fig. 1 with $(a, b) = (0.4, 0.15)$. These arise as singular perturba-

[1] In [22], the family $F(r) = b + \tanh((1 - r)a)$ was shown to also generate a wide variety of steady states, many similar to what is shown in Fig. 1.

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