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Nonlinear modulational stability of periodic traveling-wave solutions of the generalized Kuramoto–Sivashinsky equation

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HIGHLIGHTS

- Rigorous proof of nonlinear stability of spectrally stable gKS periodic waves.
- Numerical investigation of spectral stability, including rigorous error bounds.
- Numerical investigation of connection between long-time dynamics/Whitham equations.
- Thorough survey of existence theory for gKS periodic waves.
- Appendix applying nonlinear theory to a simpler nonconservative Swift–Hohenberg equation.

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ABSTRACT

In this paper we consider the spectral and nonlinear stabilities of periodic traveling wave solutions of a generalized Kuramoto–Sivashinsky equation. In particular, we resolve the long-standing question of nonlinear modulational stability by demonstrating that spectrally stable waves are nonlinearly stable when subject to small localized (integrable) perturbations. Our analysis is based upon detailed estimates of the linearized solution operator, which are complicated by the fact that the (necessarily essential) spectrum of the associated linearization intersects the imaginary axis at the origin. We carry out a numerical Evans function study of the spectral problem and find bands of spectrally stable periodic traveling waves, in close agreement with previous numerical studies of Frisch–She–Thual, Bar–Nepomnyashchy, Chang–Demekhin–Kopelevich, and others carried out by other techniques. We also compare predictions of the associated Whitham modulation equations, which formally describe the dynamics of weak large scale perturbations of a periodic wave train, with numerical time evolution studies, demonstrating their effectiveness at a practical level. For the reader's convenience, we include in an appendix the corresponding treatment of the Swift–Hohenberg equation, a nonconservative counterpart of the generalized Kuramoto–Sivashinsky equation for which the nonlinear stability analysis is considerably simpler, together with numerical Evans function analyses extending spectral stability analyses of Mielke and Schneider.

1. Introduction

Localized coherent structures such as solitary waves play an essential role as elementary processes in nonlinear phenomena. Examples of this are multi-bump solutions in reaction–diffusion equations, which are constructed by piecing together well-separated solitary waves [1], or the limiting case of infinite, periodic wave trains. A similar situation occurs in nonlinear dispersive media described by a Korteweg–de Vries (KdV) equation where exact multi-bump and periodic solutions exist. In this paper, we consider *periodic solutions* of an *unstable dissipative–dispersive* nonlinear equation, namely a generalized Kuramoto–Sivashinsky (gKS) equation

$$u_t + \gamma\partial_x^4 u + \varepsilon\partial_x^3 u + \delta\partial_x^2 u + \partial_x f(u) = 0, \quad \gamma, \delta > 0, \tag{1.1}$$

where $f(u)$ is an appropriate nonlinearity and $\varepsilon, \gamma \in \mathbb{R}$ are arbitrary constants with $\gamma > 0$. In the case $f(u) = \frac{u^2}{2}$, Eq. (1.1)

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is a canonical model for pattern formation that has been used to describe, variously, plasma instabilities, flame front propagation, turbulence in reaction–diffusion systems and nonlinear waves in fluid mechanics [2–7].

Eq. (1.1) may be derived formally either from shallow water equations [8] or from the full Navier–Stokes system [9] for $0 < \gamma = \delta \ll 1$. Here δ measures the deviation of the Reynolds number from the critical Reynolds number above which large scale weak perturbations are spectrally unstable. For this latter application, what we have in mind is the description of chaotic motions in thin film flows down an incline [6]. Indeed, periodic traveling waves are some of the few simple solutions in the attractor for the classic ($\varepsilon = 0$) Kuramoto–Sivashinsky equation, a generic equation for chaotic dynamics, and there is now a substantial amount of literature on these solutions (and their bifurcations, in particular period doubling cascades) and their stability; see [10,11,4,6]. As ε increases, the set of stable periodic waves, and presumably also their basin of attraction appears (numerically) to enlarge [12,13], until, in the $|(\gamma, \delta)| \to 0$ limit, they and other approximate superpositions of solitary waves appear to dominate asymptotic behavior [6,7,14,15].

Since $\delta > 0$ it is easily seen via Fourier analysis that all constant solutions of (1.1) are unstable, from which it follows that all asymptotically constant solutions (such as the solitary waves) are also unstable. Nevertheless, one can still construct multi-bump solutions to (gKS) on asymptotically large time $O(\delta^{-1})$ by gluing together solitary waves, provided that the distance between them is not too large [7]. One possible interpretation of this is that there exist stable periodic wave trains nearby the solitary wave. Indeed, it has been known, almost since the introduction of the classical Kuramoto–Sivashinsky equation (1.1) ($\varepsilon = 0$) in 1975, that there exists a spectrally stable band of periodic solutions in parameter space; see for example the numerical studies in [16,10]. These stable periodic wave trains may be heuristically viewed as a superposition of infinitely many well separated solitary waves. In [17], the existence of such a band of stable periodic traveling waves was justified analytically for the Eq. (1.1) with *periodic boundary conditions* in the singular KdV limit $|(\gamma, \delta)| \to 0$.

Although numerical time-evolution experiments suggest that these spectrally stable waves are nonlinearly stable as well (see [6]), up to now this conjecture had not been rigorously verified, and indeed *the standard techniques developed in, e.g.,* [18–20] *for the study of stability of periodic waves do not appear to apply*. In this paper, utilizing heavily the recent infusion of new tools in [21,22,14,23,15] in the context of general conservation laws and the St. Venant (shallow water) equations, we prove the result, previously announced in [24], that *spectral modulational stability of periodic solutions of* (1.1), defined in the standard sense, *implies linear and nonlinear modulational stability to small localized* (integrable) *perturbations*; that is, a localized perturbation of a periodic traveling wave converges to a periodic traveling wave that is modulated in phase. The first such nonlinear result for any version of (1.1), that is, for any $\gamma, \delta > 0$, this closes in particular the 35-year old open question of nonlinear stability of spectrally stable periodic waves of the classical Kuramoto–Sivashinsky equation ($\varepsilon = 0$) found numerically in [10].

With these improvements in nonlinear theory, we find this also an opportune moment to make a definitive discussion of the generalized Kuramoto–Sivashinsky equation (and Swift–Hohenberg equations) in terms of existence, nonlinear theory, and numerical spectral stability studies, all three, across all parameters, both connecting to and greatly generalizing the variety of prior works [10,11,4,6,25,26,18,12,13]. We thus carry out also a numerical analysis of the spectrum in order to check the spectral assumptions made in our main theorem.

Notice that translational invariance of the governing equations implies that the origin is always an element of the spectrum, hence that spectral stability can be at best of marginal type; this is standard for periodic solutions of equations not depending on the spatial variable x, as pointed out early on by Schneider and others [18]. Moreover, here, the conservative form of the equations introduces an additional critical mode beyond the usual translational one, a circumstance that greatly complicates both the analytical and numerical stability theory regarding the periodic wave trains admitted by (1.1). In particular, the renormalization techniques introduced in [18] and refined in, e.g., [19,20], until recently the only techniques available to treat nonlinear modulational stability of general periodic waves,[1] *do not appear to apply to situations, as here, involving critical modes with differing linear group velocities.*[2] Likewise, bifurcation of these multiple modes as Floquet number is varied around zero is a substantially more sensitive problem than bifurcation of a simple eigenvalue, making the numerical spectral stability problem more difficult as well.

Our numerical approach is based on complementary tools; namely Hills method and the Evans function. On the one hand, we use SpectrUW numerical software [29] based on Hills method, which is a Galerkin method of approximation, in order to obtain a good overview of location of the spectrum: the periodic coefficients and eigenvectors are expanded using Fourier series, and then a frequency cutoff is used to reduce the problem to finding eigenvalues of a finite dimensional matrix. It is known that Hills method converges faster than any algebraic order [30]; moreover, in practice, it gives quickly a reliable global qualitative picture. However, the associated error bounds are of abstract nature, with coefficients whose size is not a priori guaranteed. Further, near the critical zone around $\lambda = 0$, the resolution of this method is not in practice sufficient to guess at stability, let alone obtain satisfactory numerical verification.

Thus, in order to get more reliable pictures near the origin and guarantee the spectral stability of periodic wave, we use on the other hand an approach based on the Evans function of Gardner, computing a winding number to prove that there is or is not unstable spectrum in the part of the unstable (positive real part) complex half plane excluding a small neighborhood of the origin, then using Cauchy's integral formula to determine the Taylor expansions of the spectral curves passing through the origin. This method, though cumbersome for approximating global spectrum, is excellent for excluding the existence of spectra on a given region, and comes with error bounds that can in a practical sense be prescribed via the tolerance of the Runge–Kutta 4–5 scheme used to evaluate the Evans function by solution of an appropriate ODE; see [31,32,23] for general discussion of convergence of Evans function methods. Furthermore, under generic assumptions, the numerical protocol introduced in Section 2.1.3 below detects sideband stability and instability of the underlying periodic wave train without the need of lengthy spectral perturbation expansion calculations, thus adding what we believe is a valuable new method to the numerical toolbox for analyzing the spectral stability of periodic wave trains. It should be noted that there exist explicit relationships between Hill's method and Gardner's Evans function; the interested reader is encouraged to consult [30,33] and references therein.

In order to validate our numerical method, we compare our results with those known for the Kuramoto–Sivashinsky equation ($\varepsilon = 0, f(u) = u^2/2$) and for the Swift–Hohenberg equation. We see that we obtain very good agreement with several existing

[1] For Ginzburg–Landau type equations in which phase and amplitude can be approximately decoupled, there have also been used direct $L^p \to L^q$ estimates on the amplitude equations; see [27,28].

[2] Defined in Remark 1.2.