Physica D 256–257 (2013) 1–6

Contents lists available at SciVerse ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Do nonlinear waves in random media follow nonlinear diffusion equations?

T.V. Laptyeva [a,*], J.D. Bodyfelt [a], S. Flach [a,b]

[a] Max-Planck-Institut für Physik Komplexer Systeme, Dresden, Germany

[b] New Zealand Institute for Advanced Study, Massey University, Auckland, New Zealand

HIGHLIGHTS

- We study the evolution of NDE and dynamics of nonlinear disordered lattices (KG/DNLS).
- We used two key quantities: the statistical measures of second moment and kurtosis.
- The numerics show good correspondence to NDE analytics in a wide parameters range.
- We also introduced a modified NDE with long-range exponentially decaying coupling.
- Numerics for above model show even deeper correspondence of KG/DNLS and the NDE.

ARTICLE INFO

Article history:
Received 26 June 2012
Received in revised form
4 April 2013
Accepted 7 April 2013
Available online 25 April 2013
Communicated by A. Pikovsky

Keywords:
Wave localization
Nonlinear dynamics
Chaos
Wave propagation
Subdiffusion
Nonlinear diffusion equation

ABSTRACT

Probably yes, since we find a striking similarity in the spatio-temporal evolution of nonlinear diffusion equations and wave packet spreading in generic nonlinear disordered lattices, including self-similarity and scaling. We discuss, analyze and compare nonlinear diffusion equations with compact or exponentially decaying interactions, and generalized dependences of the diffusion coefficient on the density. Our results strongly support applicability to wave packet spreading in disordered nonlinear lattices.

1. Introduction

The combined impact of disorder and nonlinearity strongly affects the transport properties of many physical systems leading to complex behavior contrary to their separate linear counterparts. The application has great range; particularly relevant are nonlinear effects in cold atoms [1,2], superconductors [3], and optical lattices [4–6]. Yet experimental probing of both disordered and nonlinear media remains limited due to reachable time or size scales.

Significant achievements towards understanding the interplay of disorder and nonlinearity have been made in recent theoretical and numerical studies. A highly challenging problem was the dynamics of compact wave packets expanding in a disordered potential, in the presence of nonlinearity. The majority of studies focused on two paradigmatic models – the discrete nonlinear Schrödinger (DNLS) and the Klein–Gordon (KG) equations – revealing both destruction of an initial packet localization and its resulting subdiffusive spreading, however with debate regarding the asymptotic spreading behaviors [7–14]. Hypotheses of an ultimate slowing-down [15,16] or eventual blockage of spreading [17,18] have been recently challenged with evidence in [19], which reported a finite probability of unlimited packet expansion, even for small nonlinearities. For more details on ongoing controversial debates, we refer the reader to the recent review [20]. A qualitative theory of the nonlinear wave evolution in disordered media is based on the random phase ansatz [12], derives power-law dependences of the diffusion coefficient on the local densities, and predicts several distinct regimes of subdiffusion that match numerics [11–14] convincingly. Closely tied to these phenomena is thermal conductivity in a disordered quartic KG chain, analyzed in [21].

Similar power-law dependences of the diffusion coefficient on the local density have been extensively studied in the context of the nonlinear diffusion equation (NDE). The NDE universally

* Corresponding author.
E-mail address: lapteva@pks.mpg.de (T.V. Laptyeva).

0167-2789/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.physd.2013.04.005

describes a diverse range of different phenomena, such as heat transfer, fluid flow or diffusion. It applies to gas flow through porous media [22,23], groundwater infiltration [24,25], or heat transfer in plasma [26]. As a key trait, the NDE admits self-similar solutions (also known as the source-type solution, ZKB solution or Barenblatt–Pattle solution). It describes the diffusion from a compact initial spot and was first studied by Zel'dovich, Kompaneets, and Barenblatt [27,28].

The connection between nonlinear disordered spatial wave equations and NDE was conjectured recently and remains an open terrain [29,15,30–33]. A particularly challenging question is whether the NDE self-similar solution is an asymptotic time limit for the wave packet spreading in nonlinear disordered arrays. If yes, this will support the expectations that compact wave packets spread indefinitely, without re-entering Anderson localization. In this paper, we demonstrate that the NDE captures essential features of energy/norm diffusion in nonlinear disordered lattices. At present, we still lack a rigorous derivation of the NDE from the Hamiltonian equations for nonlinear disordered chains. Here we show that at a sufficiently large time the properties of the NDE self-similar solution reasonably approximate those of the energy/norm density distribution of nonlinear waves; manifesting in similar asymptotical behaviors of statistical measures (such as distribution moments and kurtosis), and in the overall scaling of the density profiles. To substantiate our conclusions, we perform simulations of a modified NDE and compare the results against the spatio-temporal evolution of nonlinear disordered media [13,14].

2. Theoretical predictions

2.1. Basic nonlinear disordered models

The spreading of wave packets has been extensively studied within the framework of KG/DNLS arrays. Particularly, the DNLS describes the wave dynamics in various experimental contexts, from optical wave-guides [5,6] to Bose–Einstein condensates [34]. It was found that the KG equation approximates well the DNLS one under appropriate conditions of small energy densities. This is substantiated by previous derivations of the correspondence in the ordered lattice case [35,36]. While a similar derivation for the disordered case is missing, an enormous amount of numerical data shows that the analogy is working for the spreading characteristics of wave packets [10,11,13,14]. We perform computations exactly in the same parameter regimes covered by these previous studies. Note also that the KG has the advantage of faster integration at the same level of accuracy.

The DNLS chain is described by the equations of motion

$$i\dot{\psi}_l = \epsilon_l \psi_l + \beta\, |\psi_l|^2\, \psi_l - \psi_{l+1} - \psi_{l-1}, \tag{1}$$

where ϵ_l is the potential strength on the site l, chosen uniformly from an uncorrelated random distribution $[-W/2, W/2]$ parameterized by the disorder strength W.

The KG lattice is determined by

$$\ddot{u}_l = -\tilde{\epsilon}_l u_l - u_l^3 + \frac{1}{W}(u_{l+1} + u_{l-1} - 2u_l), \tag{2}$$

where u_l and p_l are, respectively, the generalized coordinate/momentum on the site l with an energy density E_l. The disordered potential strengths $\tilde{\epsilon}_l$ are chosen uniformly from the random distribution $[1/2, 3/2]$. The total energy $\bar{E} = \sum_l E_l$ acts as the nonlinear control parameter, analogous to β in DNLS (see e.g. [11]).

Both models conserve the total energy, the DNLS additionally conserves the total norm $S = \sum_l |\psi_l|^2$. The approximate mapping from the KG to the DNLS is $\beta S \approx 3W\bar{E}$ was empirically confirmed in a large number of extensive numerical simulations [10–14]. Therefore we restrict analytics to the DNLS model. We also note that we exclude here numerical considerations for strong nonlinearities where self trapping occurs in the DNLS model rigorously due to the two integrals of motion [17]. For the KG a similar theorem does not exist (note however that again previous numerical investigations [10–14] showed that self trapping occurs in the KG case as well up to the largest computed times).

2.1.1. Spreading predictions

In order to quantitatively characterize the wave-packet spreading in Eqs. (1) and (2) and compare the outcome to the NDE model, we track the probability at the l-th site, $\mathcal{P}_l \equiv n_l = |\psi_l|^2$, where n_l is the norm density distribution. Note that the analog of n_l in the KG is the normalized energy density distribution E_l. We then track a normalized probability density distribution, $z_l \equiv n_l/\sum_k n_k$. In order to probe the spreading, we compute the time-dependent moments $m_\eta = \sum_l z_l(l-\bar{l})^\eta$, where $\bar{l} = \sum_l l z_l$ gives the distribution center.

We further use as an additional dynamical measure the kurtosis [37], defined as $\gamma(t) = m_4(t)/m_2^2(t) - 3$. Kurtosis is an indicator of the overall shape of the probability distribution profile—in particular, as a deviation measure from the normal profile. Large values correspond to profiles with sharp peaks and long extending tails. Low values are obtained for profiles with rounded/flattened peaks and steeper tails. As an example, the Laplace distribution has $\gamma = 3$, while a compact uniform distribution has $\gamma = -1.2$.

The time dependence of the second moment m_2 of the above distributions z_l was previously derived and studied in [10–14]. Different regimes of energy/norm subdiffusion were observed and explained. Generally, m_2 follows a power-law t^α with $\alpha < 1$. Here we briefly recall the key arguments.

In the linear limit Eqs. (1) and (2) reduce to the same eigenvalue problem [10,11]. We can thus determine the normalized eigenvectors $A_{\nu,l}$ and the eigenvalues λ_ν. With $\psi_l = \sum_\nu A_{\nu,l}\phi_\nu$, Eq. (1) reads in an eigenstate basis as

$$i\dot{\phi}_\nu = \lambda_\nu \phi_\nu + \beta \sum_{\nu_1,\nu_2,\nu_3} I_{\nu,\nu_1,\nu_2,\nu_3} \phi^*_{\nu_1}\phi_{\nu_2}\phi_{\nu_3}, \tag{3}$$

where $I_{\nu,\nu_1,\nu_2,\nu_3} = \sum_l A_{\nu,l}A_{\nu_1,l}A_{\nu_2,l}A_{\nu_3,l}$ are overlap integrals and ϕ_ν determine the complex time-dependent amplitudes of the eigenstates.

In [12] the incoherent "heating" of cold exterior by the packet has been established as the most probable mechanism of spreading. Following this analysis, the packet modes $\phi_\nu(t)$ should evolve chaotically with a continuous frequency spectrum. In particular, chaotic dynamics of phases is expected to destroy localization. The degree of chaos is linked to the number of resonances, whose probability becomes an essential measure for the spreading. Previous studies [38] indicate that the probability of a packet eigenstate to be resonant is $\mathcal{R}(\beta n) = 1 - e^{-C\beta n}$, with C being a constant dependent on the strength of disorder. The heating of an exterior mode close to the edge of the wave packet with norm density n would then follow

$$i\dot{\phi}_\mu = \lambda_\mu \phi_\mu + \beta n^{3/2}\mathcal{R}(\beta n) f(t) \tag{4}$$

with delta-correlated (or, reasonably, short-time correlated) noise $f(t)$, and lead to $\left|\phi_\mu\right|^2 \sim \beta^2 n^3(\mathcal{R}(\beta n))^2 t$. The momentary diffusion rate follows as $D \sim \beta^2 n^2(\mathcal{R}(\beta n))^2$.

With $m_2 \sim n^{-2} = Dt$ one arrives at $1/n^2 \sim \beta(1 - e^{-C\beta n})t^{1/2}$. Depending on the product $C\beta n$ being larger or smaller than one, the packet has two regimes of subdiffusion (and a dynamical crossover between them): $m_2 \sim \beta t^{1/2}$ (strong chaos) and $m_2 \sim \beta^{4/3}t^{1/3}$ (asymptotic weak chaos) [10–14].

The validity of the assumption of incoherent phases and of Eq. (4) was established through numerical studies for the first time by Michaely and Fishman [39], moving the above conjecture based theories onto solid grounds.