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# The complex parameter space of a two-mode oscillator model

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## HIGHLIGHTS

- A complex parameter-space is revealed for the dynamics of a simple two-mode oscillator ensemble.
- Different types of collective responses are identified.
- The classical two-mode stochastic oscillator model is generalized.
- The possibility to obtain non-trivial synchronization in natural systems is discussed.

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## ABSTRACT

The parameter-space of a simple model that exhibits nontrivial spontaneous synchronization is thoroughly investigated. The model considers two-mode stochastic oscillators, coupled through emitted pulses by a simple optimization rule. Different types of collective responses are identified as a function of two relevant model parameters that are related to the optimization threshold and the periods of the two oscillation modes. It is shown that the investigated system exhibits partial synchronization under unexpectedly general conditions.



## 1. Introduction

Spontaneous synchronization appears in a large variety of systems in nature [1–3]. Well-known examples include biological systems such as fireflies flashing in unison [4] or crickets chirping together [5], rhythmic applause [6,7], pacemaker cells in the heart [8], the menstrual cycles of women living together [9], oscillating chemical reactions, mechanically coupled metronomes, pendulum clocks hung on the same wall, and many other systems.

Several mathematical models have been proposed to explain and describe the spontaneous synchronization phenomena in large interacting ensembles. Most of these models can be grouped into one of two broad categories that are distinguished by the nature of coupling between the oscillators: those that are based on phase coupling and those that are based on a pulse-like coupling.

The prototypical model for phase coupled oscillators is the Kuramoto model [10]. However, there are many systems in nature where one cannot define an associated periodic phase variable, thus the Kuramoto model is not a suitable description for them. In the case of systems where the interaction between oscillators is pulse-driven (such as fireflies, firing neurons, rhythmic clapping, etc.), *integrate and fire* type synchronization models are used [11–14].

A novel model that leads to synchronization in a non-trivial manner was introduced by Nikitin et al. [15]. Originally the model was inspired by the study of rhythmic applause [6,7], but it is relevant for all those complex systems where the units are oscillators with fluctuating periods, and can operate in different oscillation modes. In the simplest version of the model, the oscillation modes differ in their frequencies. Such systems are frequent in nature: a few well known examples are the ensembles of thalamocortical relay neurons [16], the unicellular alga Gonyaulax polyedra [17], or the American snowy cricket [18]. In this family of models, similarly to the integrate and fire models, the oscillators are coupled through emitted pulses. Interaction between the oscillators does not however favor synchronization

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in a direct way. Instead, the dynamics of the system is aimed to keep the average output in the system close to a desired $f^*$ threshold level. This optimal threshold level is approached by switching between the oscillation modes. Whenever the average output pulse level in the system is lower than the desired one, the oscillators are working in a higher frequency mode to increase the average output intensity per unit time. Correspondingly, when the output level is higher than $f^*$, the oscillators are working in a lower frequency mode. Synchronization appears unexpectedly, as a side effect of this optimization. Numerical studies have shown that synchronization appears only for a certain parameter range of the model [19,20]. These studies were investigating the influence of the chosen $f^*$ threshold level in the optimization dynamics. Some preliminary studies were done to investigate the influence of the randomness as well. In the case of bimodal oscillators, the effect of changing the ratio of the frequencies of the two modes was however not investigated at all in previous works, and also the phase space of the model was not mapped with sufficiently high accuracy before. In the present work we focus on exploring the behavior of the model as a function of the $f^*$ threshold level and the ratio of mode periods, and explore the phase space with a much improved accuracy. We have found that this apparently simple model of bimodal oscillators has a parameter space with a rather complex and surprisingly non-trivial structure.

## 2. The two-mode stochastic oscillator model

### 2.1. Description of the model

The basic version of the model considers an ensemble of $N$ identical bimodal, globally coupled, stochastic oscillators [15]. At any time, an oscillator can either be active, emitting a signal of strength $1/N$, or inactive, emitting no signal. Therefore the total output level of the system can vary between 0 and 1. These oscillators can be intuitively thought of as flashing units. For simplicity, from now on we shall refer to active ones as *lit* and inactive ones as being in an *unlit* or *dark* state. In accordance with this intuitive picture, the sum of the units' output levels can be thought of as the total light intensity in the system.

The units are stochastic bimodal oscillators. They can operate in two oscillation modes, one with a shorter and one with a longer period. These will be referred to as mode 1 and mode 2, respectively. The periods of the modes are random, and their mean values are denoted by $\tau_1$ and $\tau_2$. Let us define in the following the dynamics of the units in a more rigorous manner.

At the beginning of each period, the oscillators are dark. After a while, they light up and stay lit until the end of the period. A full oscillation period has a stochastic duration. For the sake of more precise mathematical description, let us consider three phases during a full oscillation period, labeled $A$, $B$ and $C$, respectively. During phase $A$ and $B$ the units are dark, while during phase $C$ they are lit. The duration of phase $A$, $\tau_A$, is a random variable drawn from the interval $[0, 2\tau^*]$ with a uniform distribution. Let us denote the mean value of $\tau_A$ by $\langle \tau_A \rangle = \tau^*$. Phase $A$ is merely a means of describing the stochasticity of the duration of an oscillation period. In this paper we shall assume that $\tau^* \ll \tau_1$. The duration of phase $B$, $\tau_B$, can have two values, $\tau_{B1}$ and $\tau_{B2}$, corresponding to the two oscillation modes. The duration of the lit phase, $\tau_C$, is fixed. The average lengths of the periods of the modes is the sum of the mean durations of these three phases: $\tau_1 = \langle \tau_A \rangle + \langle \tau_{B1} \rangle + \langle \tau_C \rangle = \tau^* + \tau_{B1} + \tau_C$ and similarly $\tau_2 = \tau^* + \tau_{B2} + \tau_C$. Since the units stay lit for a greater fraction of the short period mode than the long one, the average light intensity will be larger when the units are oscillating in the short period mode.

The coupling between the oscillators is realized through an interaction that strives to optimize the total light intensity in the system, denoted $f$. At the beginning of each period, a unit decides which mode to follow based on whether the total light intensity, $f$, is greater or smaller than a threshold level $f^*$:

- If $f \leq f^*$, the shorter period mode will be chosen. Since an oscillating unit stays lit for a greater fraction of a full period when it is operating in the short mode, this will help in increasing the average light intensity in the system.
- If $f > f^*$, the longer period mode will be chosen, reducing the average total light intensity in the system.

By this dynamic, each oscillating unit individually aims to achieve a total output intensity as close to $f^*$ as possible, based on their instantaneous measurements of the output level. As a side effect of this optimization procedure, synchronization can emerge: the total output intensity of the system becomes a periodic function and the units will flash in unison [15,19–21].

The simple model presented in the previous paragraphs differs from the original one described in [15,19] only in the distribution of the duration of the stochastic phase, $\tau_A$. In the original model, $\tau_A$ was exponentially distributed, and the behavior of the system was studied as a function of the variables $\tau^* = \langle \tau_A \rangle$ and $f^*$. The present paper focuses on the case when $\tau^* \ll \tau_1$, therefore the precise statistical distribution of $\tau_A$ does not influence the results significantly. The reason for choosing a uniform distribution for this study is that using a distribution defined on a bounded interval simplifies numerical modeling of the system. Contrarily with the previous works, in this paper the system is studied as a function of the parameter $f^*$ and the ratio of the average periods of the two oscillation modes, $\tau_2/\tau_1$.

There are several variations possible on the basic version of the model. Some of these variations have been previously shown to also lead to synchronization. In [21] a version of the model with the same duration of the dark phase and a variable duration lit phase was studied, while in [20] it was shown that synchronization emerges also when using multimodal oscillators or when the $f^*$ parameter is locally fluctuating. The lit phase can occur at the beginning or at the end of the oscillation period, leading to different behaviors. Finally, in this paper we will show that synchronization will occur even when a reversed optimization is used, that aims to achieve an output as different from the threshold $f^*$ as possible.

Three versions of the bimodal oscillator model will be considered here: *model* 1. the basic model described above with a fixed-duration lit phase, and a variable duration dark phase; *model* 2. a model with variable duration lit phase and a fixed-duration dark phase; and *model* 3. fixed-duration lit phase and variable duration dark phase with a reversed choice of the long or short modes depending on the $f^*$ value. This last case will be referred to as "anti-optimization" because the oscillators strive to achieve an output as different from $f^*$ as possible. Partial synchronization will emerge in all three cases.

### 2.2. The order parameter

We need a quantitative measure to characterize the synchronization level of the system. The order parameter used in previous studies [15,19,20] measures the periodicity level of the output signal in a tedious manner, estimating the *periodicity level*, $p$ of the global signal. When considering numerical modeling to simulate the system, the output level is computed at discrete points in time. Unfortunately the periodicity measure $p$ used in previous studies turned out not to be practical when the output signal is highly periodic and is known at discrete points only. The finite time resolution limits the precision of finding of the best period, which in turn might have a significant effect on the computed value of the periodicity level $p$. The behavior of $p$ as a function of the model parameters will no longer be characteristic of the dynamics, but will