TWO DEFINITIONS OF THE GELL-MANN CHANNELS —A COMPARATIVE ANALYSIS

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In the present paper, we compare two unequivalent definitions of the Gell-Mann channels. It turns out that both definitions coincide for qubits and for qutrits one is more restrictive than the other. In higher dimensions, there exist some constraints under which both channels describe the same dynamics. Finally, we find the time-local generators for a class of the Gell-Mann channels. We relate our results to the algebra of $\mathrm{SU}(n)$ generators.

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1. Introduction

There are two methods of describing the time evolution of open quantum systems: in the language of quantum channels and quantum master equations. Quantum channels Λ are completely positive, trace-preserving maps, where we know that a map is completely positive if and only if it can be written in the Kraus form [1],

$$\Lambda[X] = \sum_{\alpha} K_{\alpha}^{\dagger} X K_{\alpha}. \tag{1}$$

The time-dependent quantum channel, also known as the dynamical map, is the solution of the master equation

$$\frac{d}{dt}\Lambda(t) = \mathcal{L}\Lambda(t) \tag{2}$$

with the condition $\Lambda(0) = \mathbb{1}$, where $\mathcal{L}$ is the (usually time-dependent) generator of evolution. If $\mathcal{L}$ does not depend on time, then it is the generator of the quantum dynamical semigroup and can be written in the Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) form [2, 3]

$$\mathcal{L}[X] = \sum_\alpha \left(V_\alpha^\dagger X V_\alpha - \frac{1}{2}\{V_\alpha V_\alpha^\dagger, X\} \right), \tag{3}$$

where we skipped the Hamiltonian part $-i[H, X]$.

As an example of the evolution that has been well analyzed in both approaches, one can consider the dynamics of a qubit. The corresponding quantum channel is defined by

$$\Lambda_P(t)[\sigma_\alpha] = \lambda_\alpha \sigma_\alpha, \tag{4}$$

where σ_α are the Pauli matrices. It is well known that (4) is completely positive if and only if its eigenvalues satisfy the Fujiwara–Algoet condition [4],

$$|\lambda_0 \pm \lambda_3| \geq |\lambda_1 \pm \lambda_2|, \tag{5}$$

and (4) is trace-preserving for $\lambda_0 = 1$. The Pauli channel is often defined by its Kraus form,

$$\Lambda_{\mathrm{P}}(t)[X] = \sum_\alpha p_\alpha(t) \sigma_\alpha X \sigma_\alpha, \tag{6}$$

with $p_\alpha(t) \geq 0$ and $\sum_\alpha p_\alpha(t) = 1$. The time-local generator of the Pauli channel can be written in the GKSL form as

$$\mathcal{L}_{\mathrm{P}}(t)[X] = \sum_\alpha \gamma_\alpha(t) \left(\sigma_\alpha X \sigma_\alpha - X \right). \tag{7}$$

There are many generalizations of the Pauli channel to higher dimensions, like the generalized Pauli channels or the Weyl channels (for recent reviews see [5–7], and also [8, 9]).

In the present paper, we are analysing the Hermitian generalization of the Pauli channel with the (generalized) Gell-Mann matrices. Let us recall that the Gell-Mann matrices σ_{ij} are given by

$$\forall_{0 \leq i < j \leq n-1} \qquad \sigma_{ij} := e_{ij} + e_{ji}, \tag{8}$$

$$\forall_{0 \leq i < j \leq n-1} \qquad \sigma_{ji} := -i(e_{ij} - e_{ji}), \tag{9}$$

$$\forall_{0 \leq j \leq n-1} \qquad \sigma_{jj} := \sqrt{\frac{2}{j(j+1)}} \Big(\sum_{i<j} e_{ii} - j e_{jj} \Big), \tag{10}$$

$$\sigma_{00} := \sum_{j=0}^{n-1} e_{jj}. \tag{11}$$

In the next section, we are going to show that the higher-dimensional analogues of definitions (4) and (6) are not equivalent. Our main goal is to analyze and compare the two definitions and find the Fujiwara–Algoet conditions for $n \geq 2$. For a subclass of channels, we would also like to check when the time-local generator of evolution has the GKSL form. Where the time-dependence of coefficients is not explicitly stated, it it assumed that they all depend on the same $t \geq 0$. The proofs to all the theorems are presented in the appendices.