SIMPLE FRACTAL CALCULUS FROM FRACTAL ARITHMETIC

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Non-Newtonian calculus that starts with elementary non-Diophantine arithmetic operations of a Burgin type is applicable to all fractals whose cardinality is continuum. The resulting definitions of derivatives and integrals are simpler from what one finds in the more traditional literature of the subject, and they often work in the cases where the standard methods fail. As an illustration, we perform a Fourier transform of a real-valued function with Sierpiński-set domain. The resulting formalism is as simple as the usual undergraduate calculus.

Keywords: calculus on fractals, Fourier transform, arithmetic.

1. Introduction

Apparently, the first attempt of a Fourier-type analysis on fractals can be found in studies of diffusion on Sierpiński gaskets [1, 2]. A generator of the diffusion process plays there the same role as a Laplacian on a manifold, so the corresponding eigenfunction expansion may be regarded as a form of harmonic analysis. An alternative route to eigenfunction expansions on fractals is to define Laplacians or gradients more directly. Here certain approaches begin with Dirichlet forms on self-similar fractals, or one takes as a departure point discrete Laplacians and performs an appropriate limit [3–6]. Four alternative definitions of a gradient (due to Kusuoka, Kigami, Strichartz and Teplyaev) are discussed in this context in [7]. Self-similarity is typically an important technical assumption. Although Laplacians defined in the above ways cannot be regarded as second-order operators, an approach where Laplacians are indeed second order is nevertheless possible and was introduced by Fujita [8, 9] and further developed by a number of authors [10–14].

A second traditional approach to harmonic analysis on fractals comes from the notion of self-similar fractal measures. The classic result of Jorgensen and Pedersen [15] states that the method works for certain fractals, such as the quaternary Cantor set, but fails in the important case of the ternary middle-third Cantor set. Moreover, the method is inapplicable in realistic cases of non-self-similar fractals.

Quite recently we have shown [16] that a 'non-Newtonian' calculus, based on Burgin's non-Diophantine arithmetic [17–21] leads to a simple and very efficient construction of a Fourier transform on fractals of a Cantor type. Gradients and Laplacians are here, respectively, first- and second-order differential operators, and self-similarity plays no role whatsoever. There is completely no difficulty with Fourier analysis of functions mapping arbitrary Cantor sets into themselves, so Jorgensen–Pedersen-type restrictions are no longer valid. The question of Fourier analysis on fractals is important for the problem of momentum representation in quantum mechanics on fractal space-times. Another recent application of the calculus is deformation quantization with minimal length [22], and the problem of wave equations on space-times modeled by Cartesian products of different fractals [23].

The goal of the present paper is to show explicitly how to apply the non-Diophantine framework to fractals more general than the Cantor set. We explicitly perform the construction for a double cover of a Sierpiński set. Similarly to the Cantor case, self-similarity is inessential. What is important, however, is the existence of a bijection f between the fractal in question and $\mathbb{R}$.

In the Sierpiński case the bijection has a space-filling property reminiscent of Peano curves [24]. The very idea that there are links between Sierpiński-type fractals and space-filling curves is not new, and was used by Molitor *et al.* in [25] in their construction of Laplacians on fractals. However, in all other respects the approach from [25] is different from what we discuss below. The idea of employing a one-dimensional integration for finding higher-dimensional integrals is known [26], but apparently has not been used in fractal contexts so far.

Since any fractal whose cardinality is continuum can be equipped with a bijection mapping it into $\mathbb{R}$, the construction is quite universal. From a practical perspective, the only difficulty is to find the bijection explicitly, but once we have found it the remaining procedure is systematic and easy to work with.

Here, out of a multitude of possible illustrations of the formalism we have decided to discuss the case of a sine Fourier transform of a real-valued function with Sierpiński-set domain. One can directly judge applicability of the method by visually inspecting the quality of the resulting finite-term reconstruction of the signal.

2. Sierpiński set

Consider $x \in \mathbb{R}_+$ and its ternary representation $x = (t_n \dots t_0.t_{-1}t_{-2} \dots)_3$. If x has two different ternary representations, we choose the one that ends with infinitely