REMARKS ON MULTISYMPLECTIC REDUCTION

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The problem of reduction of multisymplectic manifolds by the action of Lie groups is stated and discussed, as a previous step to give a fully covariant scheme of reduction for classical field theories with symmetries.

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1. Introduction

Multisymplectic manifolds constitute one of the most used and generic geometric frameworks for classical field theories. Then, the covariant reduction of field theories by symmetries requires, as a previous step, to study the reduction of multisymplectic manifolds.

This procedure should be based on the pioneering Marsden–Weinstein technique of reduction of symplectic manifolds [32] which was applied later to many different situations such as, for instance, the reduction of autonomous and nonautonomous Hamiltonian and Lagrangian systems (regular or singular) [1, 3, 10, 17, 18, 22, 24, 37], nonholonomic systems [4, 7, 11, 27], control systems [6, 21, 33, 35, 39, 40], and in other cases (Poisson, Dirac, Euler–Poincaré, Routh and implicit reduction) [5, 16, 29–31].

In relation to the problem of reduction of classical field theories, only partial results have been achieved in the context of Lagrangian, Poisson and Euler–Poincaré reduction, and other particular situations in multisymplectic field theories [8, 13–15, 23, 26, 38, 41]. Nevertheless, the problem of establishing a complete scheme of reduction for the multisymplectic case (in the line of the Marsden–Weinstein theorem), which should give a fully covariant reduction of the theory, is still unsolved.

The aim of this letter is to review the statement of the problem and give some insights in this way. In particular, a brief discussion about Noether invariants in Lagrangian field theory illustrates these considerations.

In this paper, manifolds are real, paracompact, connected and C^∞, and maps are C^∞.

2. Multisymplectic manifolds. Actions of Lie groups

Let $\mathcal{M}$ be an m-dimensional differentiable manifold, and $\Omega \in \Omega^{k+1}(\mathcal{M})$ a differentiable form in $\mathcal{M}$ ($k+1 \leq m$). For every $x \in \mathcal{M}$, the form Ω_x establishes a correspondence $\hat{\Omega}_r(x)$ between the set of r-vectors, $\Lambda^r \mathrm{T}_x\mathcal{M}$, and the set of $(k+1-r)$-forms, $\Lambda^{k+1-r}\mathrm{T}_x^*\mathcal{M}$, as

$$\hat{\Omega}_r(x)\colon \Lambda^r \mathrm{T}_x\mathcal{M} \longrightarrow \Lambda^{k+1-r}\mathrm{T}_x^*\mathcal{M}; \qquad v \mapsto i(v)\Omega_x.$$

If v is homogeneous, $v = v_1 \wedge \ldots \wedge v_r$, then $i(v)\Omega_x = i(v_1 \wedge \ldots \wedge v_r)\Omega_x = i(v_1)\ldots i(v_r)\Omega_x$. Thus, an r-vector field $X \in \mathfrak{X}^r(\mathcal{M})$ (that is, a section of $\Lambda^r\mathrm{T}\mathcal{M}$) defines a contraction $i(X)$ of degree r of the algebra of differential forms in $\mathcal{M}$. The $(k+1)$-form Ω is 1-*nondegenerate* if $\ker\Omega := \{X \in \mathfrak{X}(\mathcal{M}) \mid \hat{\Omega}_1(x)(X) = 0; x \in \mathcal{M}\} = \{0\}$.

A couple $(\mathcal{M}, \Omega)$ is a *multisymplectic manifold* if Ω is closed and 1-nondegenerate. The degree $k+1$ of the form Ω will be called *the degree of the multisymplectic manifold*. $X \in \mathfrak{X}(\mathcal{M})$ is a *Hamiltonian vector field* if $i(X)\Omega$ is an exact k-form; that is, there exists $\zeta \in \Omega^{k-1}(\mathcal{M})$ such that

$$i(X)\Omega = d\zeta. \tag{1}$$

ζ is defined modulo closed $(k-1)$-forms. The class $\bar{\zeta} \in \Omega^{k-1}(\mathcal{M})/Z^{k-1}(\mathcal{M})$ defined by ζ is called the *Hamiltonian* for X, and every element in this class is a *Hamiltonian form* for X. Furthermore, $X \in \mathfrak{X}(\mathcal{M})$ is a *locally Hamiltonian vector field* if $i(X)\Omega$ is a closed k-form. Then, for every $x \in \mathcal{M}$, there is an open neighbourhood $W \subset \mathcal{M}$ and $\zeta \in \Omega^{k-1}(W)$ such that (1) holds on W. As above, changing $\mathcal{M}$ by W, we obtain the *Hamiltonian* for X, $\bar{\zeta} \in \Omega^{k-1}(W)/Z^{k-1}(W)$, and the local Hamiltonian forms for X.

Conversely, $\zeta \in \Omega^{k-1}(\mathcal{M})$ (resp. $\zeta \in \Omega^{k-1}(W)$) is a *Hamiltonian form* (resp. a *local Hamiltonian form*) if there exists a vector field $X_\zeta \in \mathfrak{X}(\mathcal{M})$ (resp. $X_\zeta \in \mathfrak{X}(\mathcal{M})$) such that (1) holds (resp. on W). Of course, a vector field $X \in \mathfrak{X}(\mathcal{M})$ is a locally Hamiltonian vector field if, and only if, the Lie derivative $\mathrm{L}(X)\Omega = 0$. If X, Y are locally Hamiltonian vector fields, then $[X, Y]$ is a Hamiltonian vector field with Hamiltonian form $i(X \wedge Y)\Omega$. We denote by $\mathcal{H}^{k-1}(\mathcal{M})$ the $\mathbb{R}$-vector space of Hamiltonian $(k-1)$-forms. There is a natural Lie algebra structure on $\mathcal{H}^{k-1}(\mathcal{M})$ given by (see [12, 19])

$$\{\xi, \zeta\} := -i(X_\xi)\, i(X_\zeta)\Omega.$$

DEFINITION 1. Let $\Phi\colon G \times \mathcal{M} \to \mathcal{M}$ be an action of a Lie group G on a multisymplectic manifold $(\mathcal{M}, \Omega)$. We say that Φ is a *multisymplectic action* (or also that G *acts multisymplectically* on $\mathcal{M}$ by Φ) if, for every $g \in G$, Φ_g is a multisymplectomorphism, that is, $\Phi_g^*\Omega = \Omega$. Then $\mathcal{M}$ is a *multisymplectic G-space*, or also that G is a *symmetry group* of $(\mathcal{M}, \Omega)$.