A NOTE ON THE HARMONIC OSCILLATOR GROUP, QUASI-MONOMIALITY AND ENDOMORPHISMS OF THE VECTOR SPACES

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(*Received July 26, 2017 — Revised December 7, 2017*)

This article is written with the aim to establish an equivalence between three different approaches related to special polynomial sequences. A result is proved showing the equivalence of the representation of a Lie algebra, quasi-monomiality and Lie algebra of endomorphism of vector spaces. The result is illustrated with the help of an example.

Mathematics Subject Classifications: 22E60, 08A35, 33C45, 33C50, 33C80.

Keywords: monomiality principle, harmonic oscillator Lie algebra, Lie algebra of endomorphisms, 2-variable Hermite matrix polynomials.

1. Introduction

The harmonic oscillator group

The two most important nonrelativistic systems whose Schrödinger equations can be completely solved are the hydrogen atom and the harmonic oscillator. It is known that the tractability of the hydrogen atom is related to its high degree of symmetry and the similar conclusions hold for the harmonic oscillator [13].

We start with a system in one-dimensional space. In suitable units the Hamiltonian for a spinless particle subject to a harmonic oscillator potential is

$$\mathbf{H} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{x^2}{2}. \tag{1}$$

The Hilbert space $\mathcal{H}$ consists of functions $\Psi(x)$ square-integrable on the real line. The inner product is

$$(\Psi, \Phi) = \int_{-\infty}^{\infty} \Psi(x)\overline{\Phi(x)}dx. \tag{2}$$

This work has been done under Senior Research Fellowship (Award letter No. F1-17.1/2014-15/ MANF-2014-15-MUS-UTT-34170/(SA-III/Website)) awarded to the second author by the University Grants Commission, Government of India, New Delhi.

Although the eigenvalue problem

$$\mathbf{H}\Psi = \lambda\Psi, \tag{3}$$

can be solved with special function theory, a greater insight can be achieved by adopting a formal Lie-algebraic approach.

Consider the operators

$$\mathbf{J}^{\pm} = \pm\frac{1}{\sqrt{2}}\left(\frac{d}{dx} \mp x\right), \tag{4}$$

defined on $\mathcal{H}$. It is straightforward to verify the commutation relations

$$[\mathbf{J}^3, \mathbf{J}^{\pm}] = \pm\mathbf{J}^{\pm}, \qquad [\mathbf{J}^+, \mathbf{J}^-] = -\mathbf{E}, \tag{5}$$

where $\mathbf{E}$ is the identity operator and

$$\mathbf{J}^3 = \mathbf{H}. \tag{6}$$

Furthermore, from abstract relations (5), it is easy to check that the operator

$$\mathbf{C} = \mathbf{J}^+\mathbf{J}^- - \mathbf{E}\mathbf{J}^3 = \mathbf{J}^-\mathbf{J}^+ - \mathbf{E} - \mathbf{E}\mathbf{J}^3 \tag{7}$$

commutes with $\mathbf{J}^{\pm}$ and $\mathbf{J}^3$.

The complex harmonic oscillator Lie group $G(0, 1)$ [12], consists of all 4×4 matrices of the form

$$g = \begin{pmatrix} 1 & ce^{\tau} & a & \tau \\ 0 & e^{\tau} & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad a, b, c, \tau \in \mathbb{C}, \tag{8}$$

where the group operation is matrix multiplication.

The Lie algebra $\mathcal{G}(0, 1) := L[G(0, 1)]$, can be identified with the space of 4×4 matrices of the form

$$\alpha = \begin{pmatrix} 0 & x_2 & x_4 & x_3 \\ 0 & x_3 & x_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad x_1, x_2, x_3, x_4 \in \mathbb{C}, \tag{9}$$

with Lie product $[\alpha, \beta] = \alpha\beta - \beta\alpha,\ \alpha, \beta \in L[G(0, 1)]$.

The matrices

$$\mathcal{J}^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{J}^- = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{J}^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{E} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{10}$$