ABSENCE OF A GROUND STATE FOR BOSONIC COULOMB SYSTEMS WITH CRITICAL CHARGE

YUKIMI GOTO

Graduate School of Mathematical Sciences, The University of Tokyo,
3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan
(e-mail: yukimi@ms.u-tokyo.ac.jp)

(*Received August 17, 2017 — Revised October 18, 2017*)

We consider bosonic Coulomb systems with N-particles and K static nuclei. Let E_N^Z denote the ground state energy of a bosonic molecule of the total nuclear charge Z. We prove that the system has no normalizable ground state when $E_N^{N-1} = E_{N-1}^{N-1}$.

Keywords: many-body Schrödinger operator, Coulomb system, bosonic molecules, threshold eigenvalue.

1. Introduction

We consider a molecule consisting of N particles and K fixed nuclei with positive charges $z_1, \dots, z_K > 0$ located at distinct positions $R_1, \dots, R_K \in \mathbb{R}^3$. Let $Z = \sum_{i=1}^K z_i$ be the total nuclear charge. This system is described by the Hamiltonian

$$H_N^Z = \sum_{i=1}^N \left(-\Delta_i - \sum_{j=1}^K z_j |x_i - R_j|^{-1} \right) + \sum_{1 \le i < j \le N} |x_i - x_j|^{-1} + \sum_{1 \le i < j \le K} z_i z_j |R_i - R_j|^{-1}$$

acting on $L^2(\mathbb{R}^{3N})$. Here $x_i \in \mathbb{R}^3$ and Δ_i are, respectively, the particle coordinates and the three-dimensional Laplacian with respect to the coordinate x_i. It is well known that H_N^Z is a self-adjoint operator with the domain $H^2(\mathbb{R}^{3N})$ and bounded from below. The ground state energy is defined by

$$E_N^Z = \inf \operatorname{spec} H_N^Z = \inf \left\{ \langle \psi, H_N^Z \psi \rangle : \psi \in H^1(\mathbb{R}^{3N}),\ \|\psi\|_2 = 1 \right\}$$

and if it is an eigenvalue, the corresponding eigenfunction is called the ground state. We impose no symmetry requirements on the particles for our study. That actually the unrestricted ground state wave function is the same as the bosonic (totally symmetric) one is a conclusion of [20, Section 3.2.4]. It is always the case that $E_N^Z \le E_{N-1}^Z$ (see, e.g. [16]).

According to the HVZ theorem [16], $E_N^Z < E_{N-1}^Z$ implies that there exists a ground state eigenfunction of H_N^Z. Zhislin [23] proved that $E_N^Z < E_{N-1}^Z$ for $Z > N - 1$ and it is also known that the system is not bound if for a given N the total nuclear charge becomes sufficiently small. More precisely, the system has no ground state if $N \geq 2Z + K$ [18]. This implies instability of the dianion H^{2-}. In the usual fermionic case, experimental [2] and numerical [12, 21] evidence suggests that there are no stable atomic dianions X^{2-}, i.e. fermionic atoms are not bound if $N > Z + 1$ (see, [20] for further information).

In the critical case $Z = N - 1$, it might happen either $E_N^{N-1} < E_{N-1}^{N-1}$ or $E_N^{N-1} = E_{N-1}^{N-1}$. The latter case leads to absence of anions (e.g. presumably, He^-, Be^-, etc) or, otherwise, existence of bound states having zero binding energy as well [10, 4, 8]. In this paper, we prove the ground states of bosonic molecules are delocalized when $E_N^{N-1} = E_{N-1}^{N-1}$.

THEOREM 1. *Suppose $E_N^{N-1} = E_{N-1}^{N-1}$. Then there cannot be a normalizable ground state of H_N^{N-1} in $L^2(\mathbb{R}^{3N})$.*

Thus, bosonic anions X^- fail to be stable in that case. In nature, fermionic He^- anion ($N = 3$, $Z = 2$) is unstable as also numerical computations show [7] (but a virtual state can be expected as indicated by [13]). On the other hand, Hogreve [14] proved bosonic He^- can exist as a stable anion.

For the atom with $N = 2$ electrons, the anion H^- ($Z = 1$) has a ground state [22, 6]. Moreover, for bosonic atoms it is known that $E_N^Z < E_{N-1}^Z$ for all $N \leq N_c(Z) = 1.21Z + o(Z)$ as $Z \to \infty$, where $N_c(Z)$ is the maximum number of particles that can be bound to a nucleus of charge Z [5, 3]. In particular, $E_N^{N-1} < E_{N-1}^{N-1}$ for N sufficiently large. As yet it is unknown if the situation $E_N^{N-1} = E_{N-1}^{N-1}$ is true for some N.

REMARK 1. Let $Z_c > 0$ be a critical value such that for $Z > Z_c$ one has $E_N^Z < E_{N-1}^Z$, and $E_N^{Z_c} = E_{N-1}^{Z_c}$. Clearly, $Z_c \leq N - 1$ by Zhislin's theorem. Our theorem corresponds to the case $Z_c = N - 1$. In the atomic situation $K = 1$, it was shown in [4, 8], that $H_N^{Z_c}$ has a ground state if $Z_c < N - 1$ and $E_{N-1}^{Z_c} < E_{N-2}^{Z_c}$ (in [8], if $Z_c \in (N - 2, N - 1)$). Furthermore, these results are also valid for the fermionic case. But our proof works only for the unconstrained (bosonic) case because the positivity of the ground state is needed.

REMARK 2. It has been shown by Gridnev [8] that for certain combinations of charges the ground state in the three-particle Coulomb system is delocalized when the energy of the system equals the bottom of the continuous spectrum.

REMARK 3. The symmetry properties of the wave functions may have an important role in the existence and absence of the bound states at threshold. For instance, we consider the two-particle Hamiltonian $H(Z) = -\Delta_x + ZV(|x|)$, where $V \in C_c^\infty(\mathbb{R}^3)$ and $V \leq 0$, $Z > 0$ (the center of mass motion is removed). There would exist Z_c such that $\inf \operatorname{spec} H(Z) \nearrow 0$ as $Z \searrow Z_c$. If the particles are bosons or boltzons (without any symmetry restrictions) then we can repeat the superharmonic argument in the proof of Theorem 1, and thus the ground state of $H(Z_c)$ is not localized. If