POTENTIAL FUNCTIONS ADMITTED BY WELL-KNOWN SPHERICALLY SYMMETRIC STATIC SPACETIMES

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Conformal groups, which also represent the point symmetries of wave and Klein–Gordon equations, give rise to interesting forms of an associated potential function. The exact nature of the relationship between the conformal groups and the wave and Klein–Gordon equations is given by a geometric condition, that we exploit to determine a variety of potentials functions. Due to the volume of potentials, we display the results in tables.

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1. Introduction

The analysis of the wave and Klein–Gordon (KG) equations on spacetime backgrounds is important in the problem of stability; it has been actively studied in the last decade [1]. In other contexts, wave and KG equations have been related to conformal algebras of pseudo-Riemannian spaces [2–4]. Due to the importance of this application, the aim of this paper is to investigate the point symmetries and potential functions of the KG equation on some particular spaces of interest. Namely, the general study is applied to three practical problems, viz., the classification of all potential functions of KG equations within the Schwarzschild, Bertotti–Robinson and Einstein spaces.

The properties of these spaces, that is, their group structures are used to solve a geometric selection rule that identifies the potential function of the KG equation. It is well established that the vector fields of a homothetic algebra coincide with the

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Lie and Noether point symmetries of a wave and KG equations. In a recent paper [5], the Noether symmetries of the well-known spherically symmetric static solutions of the Einsteins field equations were classified, and shown to possess $G4, G17, G11$ or $G6$ maximal isometry groups. Among the well-known spherically symmetric static solutions are the Schwarzschild, de Sitter, Minkowski, Bertotti–Robinson and Einstein metrics. These metrics need no introduction within the general theory of relativity, these spacetimes provide some important illustrations of the concepts of asymptotic flatness, black holes, and event horizons.

Since we suppose that the reader is familiar with the theory and properties of spacetime collineations and the point symmetries of differential equations, we shall not present the standard procedures concerning these notions. For a concise discussion of one-parameter group of conformal motions, we refer the interested reader to [6], while for a detailed text on symmetry based methods and examples, one may consult, inter alia, [7–10]. The plan of the paper follows. Section 2 provides the geometrical preliminaries concerning KG equations. In Section 3, the main results are presented. We review the Schwarzschild, Bertotti–Robinson and Einstein spaces and their properties, and finally explore their connections to the potential functions of the KG equation. Finally, in Section 4, we discuss our conclusions.

2. KG equations

The generalized KG partial differential equation is expressed as

$$\Box u + V(x^i)u = \frac{1}{\sqrt{|-g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|-g|}g^{ik}\frac{\partial u}{\partial x^k}\right) + V(x^i)u = 0, \tag{1}$$

where $\Box$ is the d'Alembertian operator. Related to this equation, in [11] the following result was proved.

THEOREM 1. *The Lie point symmetries of the Klein–Gordon equation* (1) *in a Riemannian space of dimension n are generated from the elements of the conformal algebra of the metric, as follows:*

1. *For $n > 2$ the Lie symmetry vector is*

$$X = \xi^i(x^k)\partial_i + \left(\frac{2-n}{n}\psi(x^k)u + a_0 u + b(x^k)\right)\partial_u,$$

where ξ^k is a conformal Killing vector (*CKV*) *with a conformal factor, $\psi(x^k), b(x^k)$ is a solution of* (1) *and the following condition involving the potential is fulfilled,*

$$\xi^k V_k + 2\psi V - \frac{2-n}{2}\triangle\psi = 0. \tag{2}$$

2. *For $n = 2$ the Lie symmetry vector is*

$$X = \xi^i(x^k)\partial_i + \left(a_0 u + b(x^k)\right)\partial_u,$$