GEOMETRICALLY INDUCED SPECTRAL EFFECTS IN TUBES WITH A MIXED DIRICHLET–NEUMANN BOUNDARY

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We investigate spectral properties of the Laplacian in $L^2(Q)$, where Q is a tubular region in $\mathbb{R}^3$ of a fixed cross section, and the boundary conditions combined a Dirichlet and a Neumann part. We analyze two complementary situations, when the tube is bent but not twisted, and secondly, it is twisted but not bent. In the first case we derive sufficient conditions for the presence and absence of the discrete spectrum showing, roughly speaking, that they depend on the direction in which the tube is bent. In the second case we show that a constant twist raises the threshold of the essential spectrum and a local slowndown of it gives rise to isolated eigenvalues. Furthermore, we prove that the spectral threshold moves up also under a sufficiently gentle periodic twist.

Keywords: Laplacian, Dirichlet–Neumann boundary, tube, discrete spectrum.

1. Introduction

Relations between spectral properties and geometry belong to trademark topics in mathematical physics. A particularly interesting class of problems concerns spectra of the Laplacians and related operators in tubular regions which has various applications, among others they are used to model waveguide effects in quantum systems. The turning point here was the seminal observation that ‘bending means binding’, that is that the Dirichlet Laplacian in a tube of a fixed cross section which is bent but asymptotically straight has a nonempty discrete spectrum[1]—see e.g. [10]. It inspired a long series of investigations, for a survey we refer to the monograph [14] and the bibliography there.

A nontrivial geometry can be manifested not only in the shape of the tube but also in the boundary conditions entering the definition of the Laplacian. A simple but striking example can be found in [8]: an infinite planar strip of constant width whose one boundary is Dirichlet and the other Neumann exhibits a discrete spectrum provided the Dirichlet boundary is bent 'inward' while in the opposite case the spectral threshold remains preserved. One is naturally interested whether this effect has a three-dimensional analogue. The geometry is substantially richer in this case, of course, nevertheless our first main result—see Theorems 1 and 2 below—provides an affirmative answer of a sort to this question, namely that some bending directions are favorable from the viewpoint of the discrete spectrum existence and some are not.

Another class of geometric deformations are tube twistings. In general, they act in the way opposite to bendings: to produce bound states of the Dirichlet Laplacian supported by a locally twisted tube of a noncircular cross section, an additional attractive interaction must exceed some critical strength [11]. On the other hand, a discrete spectrum may arise in a tube which is constantly twisted and the twist is locally slowed down [13]. Note that these results have a two-dimensional analogue, namely a Hardy inequality in planar strips where the Dirichlet and Neumann condition suddenly 'switch sides' [16] and the appearance of a nontrivial discrete spectrum when a sufficiently long purely Neumann segment is inserted in between [3, 9].

In the second part of the paper we examine twisted tubes with a mixed Dirichlet–Neumann boundary. We show that the effect of twisting and its local slowdowns is present again, cf. Proposition 2, now it may occur also if the tube cross section itself exhibits a rotational symmetry but the boundary conditions violate it. Furthermore, we consider a wider class of tubes where the twist is not constant along the tube but only periodic and ask whether in this case too the threshold of the essential spectrum moves up; in Theorem 3 we demonstrate this property for twists that are sufficiently gentle.

The main results of the paper indicated above, concerning the effects of bending and twisting, constant and periodic, are presented and proved in Sections 4, 5, and 6, respectively. Before coming to it, we collect in the next two preliminary sections the needed properties of the tubes and of the operators involved.

2. Preliminaries: geometry of the waveguide

Let us begin with a curve $\ell : \mathbb{R} \to \mathbb{R}^3$ that will play the role of waveguide axis supposed to be a C^3-diffeomorphism of the real axis $\mathbb{R}$ onto $\ell(\mathbb{R})$. Without loss of generality we may parametrize it by its arc length, that is, to assume that

[1] Although it is not important in the present three-dimensional context, we note this result extends to tubes in higher dimensions [7]. In other situations involving geometrically induced eigenvalues the effect may not be that robust—see, e.g. [18].