ON PEAKED PERIODIC WAVES TO THE NONLINEAR SURFACE WIND WAVES EQUATION

JIANG BO*, YOUMING ZHOU and LIU FENG

Department of Applied Mathematics, Jiangsu University of Technology,
Changzhou, Jiangsu 213001, P. R. China
(e-mail: jb@jsut.edu.cn)

(*Received November 3, 2017 — Revised December 20, 2017*)

We present a weak formulation of the nonlinear surface wind waves equation in the sense that the existence of peaked periodic waves as weak solutions is established by employing qualitative analysis for differential equations. Based on the theoretic results, we also present all possible exact explicit expressions of peaked periodic waves.

Keywords: nonlinear surface wind waves equation, qualitative analysis, peaked wave, periodic wave.

1. Introduction

In 2001, by means of a perturbative expansion method in the physical context, Manna [1] derived the following nonlinear equation

$$u_{xt} = -\frac{3g}{hc_0}u - uu_{xx} + u_x^2 \tag{1}$$

for describing asymptotic dynamics of monochromatic short surface wind waves, where $u(x,t)$ is related to the propagation speed of an unidirectional surface wave, x and t, respectively, denote the spatial and temporal variables, h is the unperturbed initial depth, g the acceleration of gravity, and c_0 the wind velocity. In the meantime, the author of [1] pointed out that Eq. (1) admits the peakon solution of the form

$$u(x,t) = -\alpha\lambda^2 \mathrm{e}^{-\left|\frac{x+\alpha\lambda^2 t}{\lambda}\right|}, \tag{2}$$

with $\alpha = -3g/hc_0$ and λ being an arbitrary constant, from which it is easy to see that the amplitude $-\lambda\alpha^2$, the velocity $\lambda\alpha^2$ and the width λ are interrelated, unlike the peakon solution of the celebrated Camassa–Holm (CH) equation [2]. More recently, via the bifurcation approach of planar vector fields, Xie et al. [3] constructed some exact explicit traveling wave solutions including smooth periodic wave solutions, compacted solitary wave solution and some unbounded wave solutions, and asserted

*Corresponding author.

that there exists no peakon solution to Eq. (1). The Lie group analysis and explicit power series solutions to Eq. (1) were discussed in [4]. However, it seems that the sense in which the function u given by (2) satisfies Eq. (1) has not been considered. It must be emphasized that for nonlinear wave equations, the peakon solution, also called peaked solitary wave solution, should be understood as a weak solution in the distributional sense since it has no continuous derivatives at crest or trough. Motivated by this, we showed in [5] that the peakon defined by (2) can exist not only as the weak traveling wave solution but as the global weak solution to Eq. (1) provided that it belongs to a suitable Sobolev space, which indicates that assertion for nonexistence of peakon solution to Eq. (1) in [3] is not correct.

As is known, in addition to the peakon solutions, the CH equation also exhibits peaked periodic wave solutions, cf. [6, 7]. In contrast to peakon, peaked periodic wave is a type of periodic traveling wave with a peak at each crest or trough, which was also called coshoidal wave by Boyd [8], periodic peakon by Lenells [9] and periodic cusp wave by Li and Liu [10]. We should point out that peaked wave solutions, no matter whether their wave type is solitary or periodic, have received increasing attention in recent years because of their potential applications in many areas of physics, see for example [11–16] and the references cited therein. So far, several methods, such as the first integral method [17], ansatz method [18], phase plane analysis method [19] and qualitative analysis method [9,20], etc., have been proposed to search for these two kinds of peaked waves. It is worth mentioning that the qualitative analysis method proposed by Lenells is an effective way to determine traveling waves with singularities including peakons, cuspons, stumpons and composite waves, from the perspective of weak solutions. For instance, a natural framework for weak solutions was used to classify all weak traveling waves of the CH equation [9].

However, whether there are the peaked periodic waves to Eq. (1) remains unknown. In the present paper, inspired by the idea of Lenells, we will give a natural weak formulation to Eq. (1) under which the existence of peaked periodic wave solutions to Eq. (1) is determined rigorously. We also obtain some exact explicit expressions of peaked periodic wave solutions to Eq. (1). To the best of authors' knowledge, all of those obtained solutions have never been reported before.

The rest of paper is organized as follows. In Section 2, we give the definition of weak solutions and the theorem of existence of peaked periodic waves in Eq. (1). In Section 3, we present all possible exact explicit peaked periodic wave solutions to Eq. (1).

2. Existence of peaked periodic waves

We begin this section with some notation. Let $C^k(\mathbb{R})$ denote the set of all k times continuously differentiable functions on the real axis. $C_0^\infty(\mathbb{R})$ represents the space of smooth functions with compact support in $\mathbb{R}$. $L_{\rm loc}^p(\mathbb{R})$ refers to the set of all functions whose restriction on any compact subset is L^p integrable. We denote, for integers $k, p \geq 1$, by $W_{\rm loc}^{k,p}(\mathbb{R})$ the space of all functions $\phi \in L_{\rm loc}^p(\mathbb{R})$ with