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GEOMETRY OF THE DISCRETE HAMILTON–JACOBI EQUATION: APPLICATIONS IN OPTIMAL CONTROL

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In this paper, we review the discrete Hamilton–Jacobi equation from a geometric point of view. In similarity with the continuous geometric Hamilton–Jacobi theory, we propose two different discrete geometric interpretations for the equation. The first approach is based on the construction of a discrete Hamilton–Jacobi equation using discrete projective flows. For it, we develop some former results on discrete Hamiltonian systems and provide a discrete equation explicitly, which matches some previous results depicted in the literature. The interest of our method is that it retrieves some already known results, but starting from a new outlook. The second approach is formulated in terms of discrete vector fields, whose definition is not straightfoward. For this, we revisit the discrete theory of mechanics by relying on the construction of discrete vector fields taken from optimal control backgrounds. From here, we reconstruct a discrete Hamilton–Jacobi equation in a novel way, and which has not been devised in the literature before.

As a last result, both interpretations are proven to be equivalent theoretically, but the numerical results differ slightly. The discrete vector field approach seems fairly more accurate concerning numerical values in the specific example that we show, that is an optimal control problem for a nonlinear system.

Keywords: discrete geometric Hamilton-Jacobi theory, discrete mechanics, optimal control, discrete flows, discrete Hamilton equations, discrete vector fields, discrete Hamilton–Jacobi equation.

1. Introduction

The discretization of differential equations is efficient on frameworks in which we cannot compute analytical solutions of the equation and then numerical methods worked upon discretizations are needed to provide approximate solutions of our differential problem.

In recent years, there has been a growing effort to set proper discrete analogues of continuous models and design numerical methods to solve them. In this paper, we are interested in optimal control problems of dynamical systems endowed with a discrete Hamiltonian system described on discrete phase spaces. Hence, if numerical methods in geometric mechanics preserve symplecticity, since we work on a phase space, they will be more efficient.

The first inklings of discrete mechanics appeared in the realm of Lagrangian mechanics [37]. The lack of a corresponding Hamiltonian theory lead to the development of discrete Hamiltonian mechanics. Since then, some works appeared on the discretization of Lagrangian and Hamiltonian systems on tangent and cotangent bundles, what lead to variational principles for dynamical systems and principles of critical action on both the tangent and cotangent bundle [18, 19, 36] as well. This gave rise to analogies between discrete and continuous symplectic forms, Legendre transformations, momentum maps and Noether's theorem [20, 22, 25]. The Hamiltonian side specially gave rise to optimal control problems by developing a discrete maximum principle that yields discrete necessary conditions for optimality. From there, discrete Hamiltonian theories became particularly useful in distributed network optimization and derivation of variational integrators [28]. These constructions rely on numerical methods that do not only preserve symplecticity but also the momentum map in the presence of symmetries. This is why the design of working numerical integrators is in vogue, since they do not necessarily preserve conservation laws. The geometry of the space is also keypoint to perform better discretizations. For this matter, it is important to rely on symmetries and invariants of the geometric space. For example, we examine conservation of energy, conservation of angular momentum, etc., when there exists a physical interpretation of the system under study.

A first important point in this work is to observe how discrete objects differ from their continuous versions by discretizing a restricted Hamiltonian of an optimal control problem according to the "left" and "right" discretization, studied in [39, 40]. A second main point is to minimize the error in our solutions for discrete systems. This is why we propose two different approaches for obtaining a discrete, geometric Hamilton–Jacobi equation that serves as a way to obtain solutions to our optimal control problem. Our interest in a Hamilton–Jacobi theory resides in the utility of this equation to integrate the system, in an equivalent way as searching for solutions of classical Newton's equations or Lagrangian and Hamiltonian dynamics. The benefit of the Hamilton–Jacobi equation is that although the system is not completely integrable, we can find conserved quantities. Hence, we are able to provide at least a particular solution [2, 16, 26, 36]. Our outlook is twofold: on one hand, we propose a discrete geometric Hamilton–Jacobi theory interpreted in terms of discrete flows. We shall use well-known results of discrete mechanics on discrete flows to reach a discrete Hamilton–Jacobi equation that retrieves the results previously depicted in [29, 39, 40]. Nonetheless, our projective flow approach proposes a new method to work out the generating function of a discrete Hamilton–Jacobi theory by searching for a discrete section of the cotangent bundle, instead of the procedure explained in [40] that is based on a canonical transformation that