# INVARIANT SOLUTIONS TO THE KHOKHLOV–ZABOLOTSKAYA SINGULAR MANIFOLD EQUATION AND THEIR APPLICATION

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(*Received March 22, 2017 — Revised October 31, 2017*)

We find the Lie algebra of point symmetries for the singular manifold equation associated with the Khokhlov–Zabolotskaya equation (KhZ), classify one-dimensional subalgebras of this algebra, and find corresponding invariant solutions. Then we use the Bäcklund transformation between the singular manifold equation and KhZ to find new exact solutions to the last equation.

**Keywords:** KhZ equation, KhZ singular manifold equation, Lax representation, invariant solutions, symmetry reductions.

## 1. Introduction

One of the most popular models in nonlinear acoustics is the Khokhlov–Zabolotskaya (KhZ) equation

$$u_{yy} = u_{tx} + u\,u_{xx} + u_x^2, \tag{1}$$

derived from the Navier-Stokes equations and aimed to describe propagation of sound beams in incompressible materials [1, 2]. In Eq. (1) the unknown function $u = u(t, x, y)$ is the deviation from the equilibrium density of a medium. It is assumed that the wave is weakly nonlinear, almost planar and propagates in an isentropic medium with small viscosity. Substituting $s_x$ for $u$ in the above equation leads to the Lin–Reissner–Tsien equation $s_{yy} = s_{tx} + s_x\,s_{xx}$, which was introduced in [3], and therefore provides the link between KhZ equation and the problem of a transonic flow of a compressed gas past a thin airfoil [4]. The 40 years long history of Khokhlov–Zabolotskaya equation has been recently commemorated in [5], where the physical phenomena which inspired the emergence of Eq. (1) and its generalizations are comprehensively discussed. The KhZ equation appeared to be useful also in geophysics (for studying seismic waves) [6] as well as in dynamics of liquid metals [7].

Due to its extensive utility, KhZ equation is constantly a subject of research. To the purpose of this paper the attempts to find exact solutions are most interesting. The symmetry-based approach leading to group-invariant solutions of Eq. (1) was undertaken by Vinogradov and Vorob′ev [8], Lychagin [9], and by Sharomet, who classified three- and two-dimensional subalgebras of its symmetry algebra and found corresponding solutions. Also, Sharomet proved that Eq. (1) has no higher symmetries. Later, Ndogmo classified the symmetry algebra of Eq. (1) into one- and two-dimensional subalgebras and analysed arising reductions [11, 12]. The symmetry algebra for Eq. (1) was computed in [13]. The paper [14] is devoted to studying solutions of the dissipative Khokhlov–Zabolotskaya equation (or the Khokhlov–Zabolotskaya–Kuznetsov equation) by using the method of nonclassical reductions. Xu studied exact solutions of Eq. (1) which are polynomial with respect to $x$ [7]. Polyanin and Zaitsev presented a review of the knowledge of the time about the solutions of KhZ equation [15].

Eq. (1) admits a Lax representation defined by the system

$$q_t = (q^2 - u)\, q_x - u_y - q\, u_x, \tag{2}$$

$$q_y = q\, q_x - u_x, \tag{3}$$

derived in [16] and then rediscovered in [17, 18]. In [19, 20] this Lax representation was used to find exact solutions to Eq. (1), while the obtained solutions are invariant with respect to contact symmetries of (1) and can be found by means of the method of symmetry reductions.

REMARK 1. The system (2), (3) can be rewritten in the form of a nonisospectral Lax representation. Namely, suppose a function $\psi(t, x, y, q)$ defines the function $q$ implicitly, that is, for a constant $c \in \mathbb{R}$ the identity $\psi(t, x, y, q(t, x, y)) \equiv c$ holds. Differentiating this identity with respect to $t, x, y$ yields the system

$$\psi_t = (q^2 - u)\, \psi_x + (u_y + q\, u_x)\, \psi_q,$$

$$\psi_y = q\, \psi_x + u_x\, \psi_q,$$

see [21, 22]. ◇

From (3) it follows that there exists a function $v(t, x, y)$ such that

$$v_x = q, \tag{4}$$

$$v_y = \tfrac{1}{2}\, q^2 - u. \tag{5}$$

Excluding $q$ from (4) and (5) gives

$$u = \tfrac{1}{2}\, v_x^2 - v_y, \tag{6}$$

while excluding $u$ yields equation

$$v_{yy} = v_{tx} + \left(\tfrac{1}{2}\, v_x^2 - v_y\right)\, v_{xx}. \tag{7}$$

In its turn this equation admits a Lax representation [23]

$$w_t = \left(\tfrac{1}{2}\, v_x^2 - v_y\right)\, w_x, \tag{8}$$

$$w_y = -v_x\, w_x. \tag{9}$$