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NONTRANSLATION INVARIANT GIBBS MEASURES FOR MODELS WITH UNCOUNTABLE SET OF SPIN VALUES ON A CAYLEY TREE

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We consider models with nearest-neighbour interactions and with the set $[0, 1]$ of spin values, on a Cayley tree of order $k \geq 1$. It is known that the "splitting Gibbs measures" of the model can be described by solutions of a nonlinear integral equation. Recently, by solving this integral equation some periodic (in particular translation invariant) splitting Gibbs measures were found. In this paper we give three constructions of new sets of nontranslation invariant splitting Gibbs measures. Our constructions are based on known solutions of the integral equation (1.5).

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1. Introduction

Let us first give necessary definitions, then explain what is the main problem; secondly we give the history of its solutions and then formulate the part of the problem which we want to solve in this paper.

A *Cayley tree* Γ^k of order $k \geq 1$ is an infinite tree, i.e. a graph without cycles, such that exactly $k+1$ edges originate from each vertex. Let $\Gamma^k = (V, L)$ where V is the set of vertices and L the set of edges.

Two vertices x and y are called *nearest neighbours* if there exists an edge $l \in L$ connecting them. We will use the notation $l = \langle x, y \rangle$.

A collection of nearest neighbour pairs $\langle x, x_1\rangle, \langle x_1, x_2\rangle, \ldots, \langle x_{d-1}, y\rangle$ is called a *path* from x to y. The *distance* $d(x, y)$ on the Cayley tree is the number of edges of the shortest path from x to y.

For a fixed $x^0 \in V$, called the *root*, we set

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \qquad V_n = \bigcup_{m=0}^{n} W_m, \qquad L_n = \{\langle x, y\rangle \in L : x, y \in V_n\}$$

and denote

$$S_k(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \qquad x \in W_n,$$

the set of *direct successors* of x on the Cayley tree of order k.

We consider models where the spin takes values in the set [0, 1], and spins are assigned to the vertices of the tree. For $A \subset V$, a *configuration* σ_A on A is an arbitrary function $\sigma_A : A \to [0,1]$. Denote by $\Omega_A = [0,1]^A$ the set of all configurations on A. We denote $\Omega = [0,1]^V$.

The *Hamiltonian* of the model is

$$H(\sigma) = -J \sum_{\langle x,y \rangle \in L} \xi_{\sigma(x)\sigma(y)}, \tag{1.1}$$

where $J \in R \setminus \{0\}$ and $\xi : (u,v) \in [0,1]^2 \to \xi_{uv} \in R$ is a given bounded, measurable function.

Let λ be the Lebesgue measure on [0, 1]. On the set of all configurations on A the *a priori* measure λ_A is introduced as the $|A|$-fold product of the measure λ, where $|A|$ denotes the cardinality of A.

We consider a standard sigma-algebra $\mathcal{B}$ of subsets of $\Omega = [0,1]^V$ generated by the measurable cylinder subsets.

A probability measure μ on $(\Omega, \mathcal{B})$ is called a *Gibbs measure* (corresponding to the Hamiltonian H) if it satisfies the DLR equation, namely for any $n = 1, 2, \ldots$ and $\sigma_n \in \Omega_{V_n}$,

$$\mu\left(\left\{\sigma \in \Omega : \sigma\big|_{V_n} = \sigma_n\right\}\right) = \int_\Omega \mu(\mathrm{d}\omega) \nu^{V_n}_{\omega|_{W_{n+1}}}(\sigma_n),$$

where

$$\nu^{V_n}_{\omega|_{W_{n+1}}}(\sigma_n) = \frac{1}{Z_n\left(\omega\big|_{W_{n+1}}\right)} \exp\left(-\beta H\left(\sigma_n \,||\, \omega\big|_{W_{n+1}}\right)\right),$$

and $\beta = \frac{1}{T}$, $T > 0$ is temperature. Furthermore, $\sigma\big|_{V_n}$ and $\omega\big|_{W_{n+1}}$ denote the restrictions of configurations $\sigma, \omega \in \Omega$ to V_n and W_{n+1}, respectively. Next, $\sigma_n : x \in V_n \mapsto \sigma_n(x)$ is a configuration in V_n and

$$H\left(\sigma_n \,||\, \omega\big|_{W_{n+1}}\right) = -J \sum_{\langle x,y \rangle \in L_n} \xi_{\sigma_n(x)\sigma_n(y)} - J \sum_{\langle x,y \rangle:\, x \in V_n, y \in W_{n+1}} \xi_{\sigma_n(x)\omega(y)}.$$

Finally,

$$Z_n\left(\omega\big|_{W_{n+1}}\right) = \int_{\Omega_{V_n}} \exp\left(-\beta H\left(\widetilde{\sigma}_n \,||\, \omega\big|_{W_{n+1}}\right)\right) \lambda_{V_n}(d\widetilde{\sigma}_n).$$

The *main problem* for a given Hamiltonian is to describe all its Gibbs measures. See [8] for a general definition of Gibbs measure, motivations why these measures are important and the theory of such measures.

This main problem is not completely solved even for simple Ising or Potts models on a Cayley tree with a finite set of spin values. Mainly this problem is solved for the class of splitting Gibbs measures (SGMs) [11] (Markov chains [8]), which are limiting Gibbs measures constructed by Kolmogorov's extension theorem of the following finite-dimensional distributions: given $n = 1, 2, \ldots,$ consider the