TOPOLOGICAL PROPERTIES OF A CURVED SPACETIME

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The present paper aims at the study of a topology on Lorentzian manifolds, defined by Göbel [4] using the ideas of Zeeman [16]. Observing that on the Minkowski space it is the same as Zeeman's time topology, it has been found that a Lorentzian manifold with this topology is path connected, nonfirst countable and nonsimply connected while the Minkowski space with time topology is, in addition nonregular and separable. Furthermore, using the notion of Zeno sequences it is obtained that a compact set does not contain a nonempty open set and that a set is compact if and only if each of its infinite subsets has a limit point if and only if each of its sequences has a convergent subsequence.

Keywords: Lorentzian manifold, Minkowski space, non-manifold topologies, time topology, regularity, path connectedness, simple connectedness.

1. Introduction

Zeeman's [16] path breaking ideas of defining topologies on flat and curved spacetimes taking into account the causal structure of spacetime, paved the way for the use of topology in general relativity. The homeomorphism group of Lorentzian manifolds with these topologies was found to be the conformal group, thus making these topologies physically significant [4, 5, 7, 12, 11]. One such topology is the path topology defined by Hawking et al. [7] which has been extensively studied for its topological properties, classification problem, compact sets and simple connectedness by Hawking et al. [7], Agrawal et al. [1, 2] and Low [9]. Other studies include the study of singularities, hypersurfaces, rotation matrix of spacetime [3, 6, 8, 14, 13], etc.

The study undertaken in the present paper includes the study of topological invariants of Lorentzian manifolds with a topology defined by Göbel [4] using the ideas of Zeeman [16]: topological invariants are well-known tools used in the classification of topological spaces used to determine topological phases of matter by David J. Thouless, F. Duncan, M. Haldane and J. Michael Kosterlitz for which they have been recently awarded the 2016 Nobel Prize in Physics. We call it the geodesic topology. To understand this topology, we begin with the following prerequisites.

Throughout this paper, $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Q}$ denote the set of real, natural and rational numbers, respectively. For $n \in \mathbb{N}$ and $n > 1$, the bilinear form $g : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$, satisfying the following properties: (i) symmetric, i.e. for all $x, y \in \mathbb{R}^n$, $g(x, y) = g(y, x)$, (ii) nondegenerate, i.e. if for all $y \in \mathbb{R}^n, g(x, y) = 0$, then $x = 0$ and (iii) index is 1, i.e. there exists an ordered basis $\{e_0, e_1, \ldots, e_{n-1}\}$ for $\mathbb{R}^n$ with $g(e_i, e_j) = 1$ if $i = j = 0$; -1 if $i = j = 1, \ldots, n-1$ and 0 otherwise, is called the *Lorentzian inner product* and its matrix is known as the *Lorentzian metric*. For $x \equiv (x^0, x^1, x^2, \ldots, x^{n-1})$ in $\mathbb{R}^n$, $g(x, x) = (x^0)^2 - (x^1)^2 - \ldots - (x^{n-1})^2$, where x^0 is called the time coordinate of x and x^i's the spatial coordinates of x, for $i = 1, 2, \ldots, n-1$. A *Lorentzian manifold* L is a connected Hausdorff C^∞ real n-dimensional manifold with the Lorentzian metric. In particular, $\mathbb{R}^n$ with the Lorentzian metric is called the n-dimensional Minkowski space and is denoted by M.

The Lorentzian metric divides the elements of the tangent space $T_p(L)$, $p \in L$ into three classes: a vector $X \in T_p(L)$ is said to be timelike, lightlike or spacelike if $g(X, X)$ is positive, zero or negative, respectively. The map $\exp_p : T_p(L) \longrightarrow L$ such that $\exp_p(X_p) = \gamma(1)$ is called the *exponential map*, where γ is the unique geodesic passing through p along the tangent vector X_p. There exists a neighbourhood N of the origin of $T_p(L)$ such that $\exp_p : N \longrightarrow U$, where $U \equiv \exp_p(N)$, is a diffeomorphism: U is called a normal neighbourhood of p. The coordinates of each point $q \in U$ are the coordinates of $\exp_p^{-1}(q) \equiv X_q \in N$ with respect to some orthonormal basis of $T_p(L)$. These coordinates are called the *normal coordinates* of $q \in U$. The set $C^T(p, U) = \{q \in U : \exp_p^{-1}(q)$ is a timelike vector and $p = q\}$ is called the *time cone* at p.

Geodesic topology, the object of study of the present paper, is defined as the finest topology that induces manifold topology on every timelike geodesic: the terminology 'timelike geodesic topology' would have been more explanatory, but as it is too lengthy, we prefer to call it the 'geodesic topology'. Being finer than the manifold topology, it is Hausdorff and noncompact and as every timelike straight