A DIVERGENCE THEOREM FOR PSEUDO-FINSLER SPACES

E. Minguzzi

Dipartimento di Matematica e Informatica "U. Dini", Università degli Studi di Firenze, Via S. Marta 3, I-50139 Firenze, Italy
(e-mail: ettore.minguzzi@unifi.it)

(*Received January 4, 2017 – Revised March 31, 2017*)

We study the divergence theorem on pseudo-Finsler spaces and obtain a completely Finslerian version for spaces having a vanishing mean Cartan torsion. This result helps to clarify the problem of energy-momentum conservation in Finsler gravity theories.

Keywords: Finsler gravity, Finslerian relativity, conservation laws.

1. Introduction

Two Finslerian divergence theorems have been discussed by H. Rund [1] and Z. Shen [2, 3]. While they both equate a certain volume integral on a domain of M with a surface integral at the boundary, they have quite a different nature.

Let $\pi\colon TM \to M$ be the tangent bundle, let $E = TM\backslash 0$ be the slit tangent bundle, and let VE be the kernel of π_*, namely the vertical tangent bundle. Let $\mathscr{L}\colon TM\backslash 0 \to \mathbb{R}$ be the Finsler Lagrangian (for detailed definitions we refer the reader to Section 2), and let g be its vertical Hessian, namely the Finsler metric. Rund's theorem involves a vector field $Z\colon E \to TM$, a section $s\colon M \to E$, and the Finslerian divergence of Z on E calculated with the horizontal covariant derivative and pulled back to M using s (cf. Eq. (4)). It has the drawback that the boundary term is not genuinely Finslerian, indeed, both the normal to the hypersurface and the boundary form are deduced from the pullback metric s^*g. So there appear elements of Riemannian geometry.

The version by Shen is somewhat complementary [3, Theorem 2.4.2]. It is less Finslerian for what concerns the vector field since it deals with a field $X\colon M \to TM$ on the base, for the computation of whose divergence the Finsler connection is not required. However, the boundary term is genuinely Finslerian as it involves the notion of Finsler normal to a hypersurface.

In this work we are going to elaborate a further version of the divergence theorem in Finsler geometry. First, we shall give a short derivation of Rund's result and then we shall show that for pseudo-Finsler spaces with vanishing mean Cartan torsion, $I_\alpha = 0$, it is possible to give a genuinely Finslerian divergence theorem in which both the vector field and the boundary terms are Finslerian.

It must be recalled that by Deicke's theorem [4–6], a Finsler space with vanishing mean Cartan torsion is actually Riemannian. As a consequence, the mentioned result will be of interest only for pseudo-Finsler spaces of nondefinite signature. It has been suggested by the author that Lorentz–Finsler spaces with zero mean Cartan torsion might be the appropriate objects of study in Finsler gravity theory [7–9], particularly because they have affine sphere indicatrices and because, as in general relativity, they are uniquely determined by a spacetime volume form and a light cone distribution [10]. Therefore, it is expected that the results of this work could shed light on the problem of energy-momentum conservation in Finslerian extensions of general relativity.

2. Connections in pseudo-Finsler geometry

In this section we assume some familiarity with the notion of Finsler connection and of pullback connection [7, 11–16]. Let us give some key coordinate expressions in order to fix terminology and notation. Let $\{x^\mu\}$ be coordinates on a chart of M and let $\{x^\mu, y^\mu\}$ be the induced coordinates on TM. The Finsler Lagrangian is, by definition, positive homogeneous of degree two $\mathscr{L}(x, sy) = s^2\mathscr{L}(x, y)$, $\forall s > 0$. Although we assumed that $\mathscr{L}$ is defined on the slit tangent bundle, this assumption can be relaxed, e.g. it could be defined on just a convex cone subbundle provided the next equations are evaluated on its domain, see [17, 18] for a complete discussion. The Finsler metric is given by the Hessian $g_{\mu\nu} = \partial^2\mathscr{L}/\partial y^\mu \partial y^\nu$ and is assumed nondegenerate. The Cartan torsion is

$$C_{\alpha\beta\gamma} = \frac{1}{2}\frac{\partial}{\partial y^\alpha} g_{\beta\gamma}$$

while the mean Cartan torsion is $I_\gamma = g^{\alpha\beta}C_{\alpha\beta\gamma}$.

EXAMPLE 1. A nontrivial example of affine sphere spacetime is

$$\mathscr{L} = -\tfrac{2}{3^{3/4}}\left(\left(\tfrac{1}{2}dt + \tfrac{\sqrt{3}}{2}a(t)dz\right)^2\right)^{1/4}\left(\left(\tfrac{\sqrt{3}}{2}dt - \tfrac{1}{2}a(t)dz\right)^2 - a^2(t)(dx^2 + dy^2)\right)^{3/4},$$

which in the low velocity limit (with respect to an observer whose velocity field is $u \propto \partial_t$) gives the general relativistic Friedmann metric in the flat space section case (i.e. $k = 0$) $g = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$. Here $a(t)$ is the scale factor of the Universe. Many other examples of affine sphere spacetimes and a discussion can be found in [9]. There are really plenty of affine sphere spacetimes since there are plenty of cone structures and volume forms. If the light cones have ellipsoidal sections one recovers the usual Lorentzian spacetimes [10]. We remark that in what follows we do not assume a Lorentzian signature for the Finsler metric, though this is certainly the most interesting case.

A nonlinear connection is a splitting of the tangent space $TE = VE \oplus HE$ into vertical and horizontal bundles. A basis for the horizontal space is given by