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# COHERENCE AND SQUEEZING ALONG QUANTUM TRAJECTORIES

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We perform a detailed analysis of the behaviour of coherent and squeezed states undergoing the time evolution. We calculate the time dependence of expectation values of position and momentum in coherent and squeezed states (which can be interpreted as quantum trajectories in coherent and squeezed states) and examine how the coherence and squeezing is affected during the time development, calculating the time dependence of the position and momentum uncertainty. We focus our investigations on two quantum systems. First we consider the quantum linear system with the Hamiltonian quadratic in $q$ and $p$ variables. As the second system we consider the simplest quantum nonlinear system with the Hamiltonian quartic in $q$ and $p$ variables. We calculate the explicit formulae for the time development of expectation values and uncertainties of position and momentum in an initial coherent state.

**Keywords:** coherent state, squeezed state, pseudo-probabilistic distribution function, quantum Hamiltonian system, harmonic oscillator.

## 1. Introduction

Coherent and squeezed states play an important role in quantum optics [1–5] (see also [6, 7] for an overview of quantum optics in phase space, and [8–10] and references therein for an extensive introduction to the topic of coherent states and their generalizations). They are, moreover, the closest analog of classical states. However, the coherence of states is a property which is not preserved during the time evolution of most quantum systems. In fact, for a physical quantum system, states undergoing the time development often finally approach one of the stationary states of the system. For this reason in quantum mechanics one is mostly dealing with stationary states (eigenvectors of the Hamilton operator). Worth noting is the paper [11] where the authors discuss the problem of stability of coherent states for particular classes of potentials.

In classical Hamiltonian mechanics solutions of Hamilton equations represent trajectories in phase space, which are an important tool in investigation of the geometry of Hamiltonian systems. The analog of classical trajectories can be also formulated in quantum mechanics [12]. In the classical case trajectories represent the time development of points in phase space (classical coherent states). However, in the quantum case points in phase space lose the physical interpretation, and thus trajectories itself are not "physical" objects. Nevertheless, quantum trajectories can be used to calculate the time dependence of expectation values of observables which have a direct physical interpretation. In fact, solutions of quantum Hamilton equations, which can be interpreted as quantum trajectories (represented in some coordinates), describe the time development of position and momentum observables. Taking expectation values of position and momentum results in a trajectory in a quantum state, which describes an average path along which a particle in phase space will move.

In this paper we would like to investigate in more detail the behaviour of coherent and squeezed states during the time development. In particular, we will calculate the time dependence of expectation values and uncertainties of position and momentum in coherent and squeezed states (quantum trajectories in coherent and squeezed states) and examine how the coherence and squeezing is affected during the time development.

The analysis will be performed for two fairly different quantum systems. The first system is a linear quantum system, described by a general Hamiltonian quadratic in $q$ and $p$ variables, so in that case quantum trajectories coincide with their classical counterparts. This is one of the simplest Hamiltonians we can consider and has the form of the Hamiltonian of the harmonic oscillator with some interaction term. Such Hamiltonian is very often found in quantum optics where it describes nonlinear interactions of light with a medium. In practice, such nonlinear effects are used to create squeezed states of light. For such Hamiltonian time evolution of expectation values of positions and momenta coincides with quantum trajectories and, as we show in Section 3, even in this case coherence is not always preserved in time.

The second system represents a nonlinear quantum system, described by a Hamiltonian quartic in $q$ and $p$ variables. It is an example of a system with the purely quantum time evolution, i.e. the solutions of quantum Hamilton equations do not coincide with their classical counterparts. This is an interesting example of a pure quantum flow, revealing the property that the evolution of such systems is not well defined for all values of the evolution parameter $t$.

Singularities of classical trajectories are not admissible as each classical trajectory represents simultaneously expectation values of positions and momenta of a system in a classical coherent state, i.e. Dirac delta distribution. Contrary, pure quantum trajectories themselves are not "physical" objects as Dirac distributions are not admissible quantum states, so, singularities of pure quantum trajectories are acceptable.

Unfortunately, even when we consider the time evolution of expectation values of position and momentum observables in a quantum coherent state, we get formulae