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A LOOMIS–SIKORSKI THEOREM AND FUNCTIONAL CALCULUS FOR A GENERALIZED HERMITIAN ALGEBRA

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A generalized Hermitian (GH-) algebra is a generalization of the partially ordered Jordan algebra of all Hermitian operators on a Hilbert space. We introduce the notion of a gh-tribe, which is a commutative GH-algebra of functions on a nonempty set X with pointwise partial order and operations, and we prove that every commutative GH-algebra is the image of a gh-tribe under a surjective GH-morphism. Using this result, we prove that each element a of a GH-algebra A corresponds to a real observable ξ_a on the σ-orthomodular lattice of projections in A and that ξ_a determines the spectral resolution of a. Also, if f is a continuous function defined on the spectrum of a, we formulate a definition of $f(a)$, thus obtaining a continuous functional calculus for A.

Keywords: synaptic algebra, generalized Hermitian algebra, order-unit space, Jordan algebra, vector lattice, MV-algebra, Loomis–Sikorski theorem, state, extremal state, observable, functional calculus.

1. Introduction

Generalized Hermitian (GH-) algebras, which were introduced in [17] and further studied in [18, 20], incorporate several important algebraic and order theoretic structures including effect algebras [15], MV-algebras [7], orthomodular lattices [33], Boolean algebras [45], and Jordan algebras [36]. Apart from their intrinsic interest, all of the latter structures host mathematical models for quantum-mechanical notions such as observables, states, properties, and experimentally testable propositions [12, 46] and thus are pertinent in regard to the quantum-mechanical theory of measurement [5].

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It turns out that GH-algebras are special cases of the more general synaptic algebras introduced in [14], and further studied in [19, 21–23, 25–28, 42]. Thus, in this paper, it will be convenient for us to treat GH-algebras as special kinds of synaptic algebras (see Section 3 below). In Section 4, we focus on commutative GH-algebras. A commutative GH-algebra A can be shown to be isomorphic to a lattice ordered Banach algebra $C(X, \mathbb{R})$, under pointwise operations and partial order, of all continuous real-valued functions on a basically disconnected compact Hausdorff space X (see Theorem 4.4).

As indicated by the title, one of our purposes in this paper is to formulate and prove an analogue for commutative GH-algebras of the classical Loomis–Sikorski representation theorem for Boolean σ-algebras (Theorem 6.6 below). See the remarks after Theorem 6.2 for an explanation of an advantage that the Loomis–Sikorski representation of a commutative GH-algebra may have over the continuous-function representation provided by Theorem 4.4.

A real observable ξ for a physical system $\mathcal{S}$ is understood to be a quantity that can be experimentally measured, and that when measured yields a result in a specified set $\mathbb{R}_\xi$ of real numbers. A state ρ for $\mathcal{S}$ assigns to ξ an expectation, i.e. the long-run average value of a sequence of independent measurements of ξ in state ρ. If f is a function defined on $\mathbb{R}_\xi$, then $f(\xi)$ is defined to be the observable that is measured by measuring ξ to obtain, say, the result $\lambda \in \mathbb{R}_\xi$, and then regarding the result of this measurement of $f(\xi)$ to be $f(\lambda)$.

In Theorem 7.4 we use our Loomis–Sikorski theorem to show that each element a in a GH-algebra A corresponds to a real observable ξ_a. In Corollary 7.5, we obtain an integral formula for the expectation of the observable ξ_a in state ρ. Definition 7.6 and the following results provide a continuous functional calculus for A.

2. Preliminaries

In this section we review some notions and some facts that will be needed as we proceed. We abbreviate 'if and only if' as 'iff,' the notation := means 'equals by definition,' $\mathbb{R}$ is the ordered field of real numbers, $\mathbb{R}^+ := \{\alpha \in \mathbb{R} : 0 \leq \alpha\}$, and $\mathbb{N} := \{1, 2, 3, ...\}$ is the well-ordered set of natural numbers.

DEFINITION 2.1. Let $\mathcal{P}$ be a partially ordered set (poset) with partial order relation $\leq$. Then:

(1) Let $p, q \in \mathcal{P}$. Then an existing supremum, i.e. least upper bound, (an existing infimum, i.e. greatest lower bound) of p and q in $\mathcal{P}$ is written as $p \vee q$ ($p \wedge q$). If it is necessary to make clear that the supremum (infimum) is calculated in $\mathcal{P}$, we write $p \vee_{\mathcal{P}} q$ ($p \wedge_{\mathcal{P}} q$). $\mathcal{P}$ is a *lattice* iff $p \vee q$ and $p \wedge q$ exist for all $p, q \in \mathcal{P}$. If $\mathcal{P}$ is a lattice, then a nonempty subset $\mathcal{Q} \subseteq \mathcal{P}$ is a *sublattice* of $\mathcal{P}$ iff, for all $p, q \in \mathcal{Q}$, $p \vee q, p \wedge q \in \mathcal{Q}$, in which case $\mathcal{Q}$ is a lattice in its own right with $p \vee_{\mathcal{Q}} q = p \vee q$ and $p \wedge_{\mathcal{Q}} q = p \wedge q$.

(2) A lattice $\mathcal{P}$ is *distributive* iff, for all $p, q, r \in \mathcal{P}$, $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$, or equivalently, $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$.