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Eigenoscillations of a fluid in a canonical domain and functional difference equations

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HIGHLIGHTS

- Explicit formulas for the eigenmodes of the continuous spectrum are considered.
- New type of the functional-difference equations are solved.
- The far field asymptotics of the eigenfunctions are obtained.
- Some physical analysis of the solution for the problem at hand is also given.

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ABSTRACT

In this work we construct and discuss special solutions of a homogeneous problem for the Laplace equation in a domain with cone-shaped boundaries. The problem at hand is interpreted as that describing oscillatory linear wave movement of a fluid under gravity in such a domain. These solutions are found in terms of the Mellin transform and by means of the reduction to some new functional-difference equations solved in an explicit form (by quadrature). The behavior of the solutions at large distances is studied by use of the saddle point technique. The corresponding eigenoscillations of a fluid are then interpreted as generalized eigenfunctions of the continuous spectrum.

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1. Introduction

Let us consider a linear time-harmonic water-wave process described by the velocity potential $U(X,Y,Z,t)$; see e.g. [1]. The complex velocity potential u is connected with the unsteady potential

$$U(X,Y,Z,t)=\Re\{\exp(-\mathrm{i}\Omega t)u(X,Y,Z)\},$$

where Ω is the angular frequency [1].

The velocity potential satisfies the Laplace equation

$$\triangle U(X,Y,Z,t)=0$$

in the domain W, see Fig. 1, $\triangle = \partial_X^2+\partial_Y^2+\partial_Z^2$, $t > 0$, the dynamic boundary condition on the free surface F $(Z=0,\ |X|>0,\ |Y|>0)$

$$U_{tt}-g\,U_Z=0,\quad t>0,$$

where g is the gravitational acceleration.

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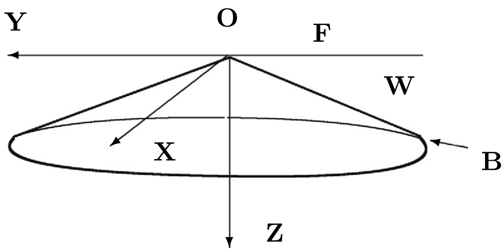

Fig. 1. Canonical domain.

The boundary condition on the conical surface of the bottom B can be taken in the form

$$U_n + \eta^{-1} U_t = 0, \quad t > 0,$$

where U_n is the derivative of the potential with respect to the normal to B directed into W. The boundary condition on the bottom requires a comment. The summand $\eta^{-1} U_t$ is responsible for the process of infiltration of fluid through the bottom, the parameter η specifies the 'velocity' of the infiltration. If the value η^{-1} turns out to be small and the effect of the infiltration is negligible, one can use the model of rigid bottom $U_n|_B= 0$. In the model of rigid bottom this term $\eta^{-1} U_t$ is absent so that formally we put $\eta^{-1} = 0$. In this work, we consider the opposite limiting case $\eta^{-1} = \infty$ of 'perfect' infiltration and assume that $U_t|_B= 0$ which is valid provided that the Dirichlet boundary condition

$$u(X, Y, Z)|_B= 0$$

is satisfied on the bottom for the stationary problem.[1] In addition, the potential is such that $U(X, Y, 0, 0)$ and $U_t(X, Y, 0, 0)$ i.e are assumed to be known at the initial moment of time $t = 0$. We may assume that the free surface is at equilibrium at the initial moment.

The domain shown in Fig. 1 might serve to simulate water-wave processes near a shallow or sandbank in the ocean. For this reason such a canonical problem may be called the linear water-wave problem for a sandbank. At the same time one can encounter similar geometry when studying the water-wave processes near a small island or atoll in the ocean.

The stationary complex potential $u(X, Y, Z)$ is governed by the Laplace equation, see the next section for the complete formulation. The boundary condition with the spectral parameter is valid on the free water surface F, whereas on the bottom B of the conical shape the Dirichlet condition is satisfied. The boundary condition on the free water surface F arises from the nonstationary one discussed above. Recall that the problem is considered in the linear small-amplitude wave approximation. By means of the Mellin integral representation we separate the radial variable. The solution is reduced to that for a problem in a domain of the unit sphere. Some new functional-difference equations are then derived and solved, which leads to a closed form representation for the corresponding eigenfunctions. By use of the saddle point technique the Mellin integral is then asymptotically evaluated at large distances from the vertex. The obtained classical solutions of the homogeneous problem, i.e. eigenoscillations, are interpreted as eigenfunctions of the continuous spectrum of infinite multiplicity.

It should be mentioned, however, that different kinds of functional equations were considered in fluid mechanics [2,3], in diffraction theory [4–12], in theoretical and mathematical physics [13,14]. It is worth remarking that the study and solution of some new functional equations is at the core of the present research. Contrary to the trigonometric coefficients for the Malyuzhinets' equations, this new type of the equations has more complex coefficients which depend on the associated Legendre functions. The equations considered in the present work are similar (but not the same) to those studied in our work [3]. Additionally, in comparison with that in [3] we give detailed the asymptotic analysis of the new eigensolutions based on the saddle point technique and discuss their physical interpretation.

2. Formulation of the problem

The Cartesian coordinates X, Y, Z are connected with the spherical ones in W by the formulas

$$X = r\cos\varphi\sin\theta, \quad Y = r\sin\varphi\sin\theta, \quad Z = r\cos\theta,$$

the axis OZ directed vertically downwards as shown in Fig. 1. Let the fluid medium occupy the conical domain, $\omega = (\theta, \varphi)$

$$W = \left\{(r, \omega): \quad r > 0, \quad -\pi < \varphi \le \pi, \quad \theta_1 < \theta < \frac{\pi}{2}\right\},$$

and $\theta = \theta_1$ is the equation of the bottom B, $0 < \theta_1 < \frac{\pi}{2}$.

[1] Although, in a sense, such classical condition might be considered artificial in the theory of linear gravity water waves, the corresponding problem is of interest from the mathematical point of view.

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