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Bidirectional Whitham equations as models of waves on shallow water

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HIGHLIGHTS

- Comparisons between bidirectional Whitham equations and water wave experiments.
- Whitham-type models are more accurate than the KdV and Serre models.

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ABSTRACT

Hammack & Segur (1978) conducted a series of surface water-wave experiments in which the evolution of long waves of depression was measured and studied. This present work compares time series from these experiments with predictions from numerical simulations of the KdV, Serre, and five unidirectional and bidirectional Whitham-type equations. These comparisons show that the most accurate predictions come from models that contain accurate reproductions of the Euler phase velocity, sufficient nonlinearity, and surface tension effects. The main goal of this paper is to determine how accurately the bidirectional Whitham equations can model data from real-world experiments of waves on shallow water. Most interestingly, the unidirectional Whitham equation including surface tension provides the most accurate predictions for these experiments. If the initial horizontal velocities are assumed to be zero (the velocities were not measured in the experiments), the three bidirectional Whitham systems examined herein provide approximations that are significantly more accurate than the KdV and Serre equations. However, they are not as accurate as predictions obtained from the unidirectional Whitham equation.

1. Introduction

Hammack & Segur [1] performed a series of tightly-controlled laboratory water-wave experiments in a long, narrow tank with relatively shallow (10 cm) undisturbed water and a wave maker at one end. The wave maker was a rectangular, vertically-moving piston located on the bottom of the tank, adjacent to a rigid wall at the upstream end of the tank. Experiments were initialized by rapidly moving the piston downward a prescribed amount that varied between experiments. This downward motion lead to the creation of initially rectangular waves wholly below the still water level, occupying the entire width and 61 cm of the upstream end of the tank. The evolution of the wave train downstream from the wave maker was investigated. Time series were collected at five gauges located 61 cm ($x = 0$, the downstream edge of the piston), 561 cm ($x = 500$), 1061 cm ($x = 1000$), 1561 cm ($x = 1500$), and 2061 cm ($x = 2000$) downstream. The tank was long enough that waves reflecting from the far end of the tank did not impact the time series collected. Among other things, Hammack & Segur showed that many analytic and asymptotic results obtained from the KdV equation compared favorably with measurements from the experiments.

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In this paper, we focus on the two experiments presented in Figures 2 and 3 of [1], which we refer to as experiment #2 and experiment #3 respectively. The experiments were identical except for the magnitude of the piston displacement and hence initial wave amplitude. In experiment #2, the piston moved downward 1 cm, producing a downstream propagating wave with an initial amplitude of 0.5 cm. In experiment #3, the piston stroke and initial amplitude were 3 cm and 1.5 cm respectively. The time series from both experiments show leading triangular waves of depression followed by series of trailing wave groups.

The main goal of this paper is to compare and evaluate a number of unidirectional and bidirectional Whitham-type equations by comparing their predictions with the experimental time series. In doing this, we demonstrate that in order to most accurately model these experimental measurements, a model must include (i) an accurate reproduction of the Euler phase velocity, (ii) sufficient nonlinearity, and (iii) surface tension effects. To our knowledge, these are the first comparisons between the recently derived bidirectional Whitham equations and data from physical experiments.

This paper is organized as follows. The model equations and their properties are presented in Section 2. Comparisons between the experimental time series and data from numerical simulations of these equations are included in Section 3. A summary is contained in Section 4

2. Model equations

The equations that describe the irrotational motion of an inviscid, incompressible, homogeneous fluid with a free surface are known as the Euler equations [2]. As the experiments of interest here were conducted in a long, narrow tank, we use two-dimensional models (i.e. models with one horizontal and one vertical dimension). The linear phase velocity for the Euler equations is given by

$$c_E = \pm\sqrt{\frac{(g+\tau k^2)\tanh(kh_0)}{k}}, \tag{1}$$

where g represents the acceleration due to gravity, τ represents the coefficient of surface tension, h_0 represents the mean depth of the fluid, and k represents the wave number of the linear wave. In all of our calculations we used $g = 981$ cm/s^2 and $\tau = 72.86$ cm^3/s^2, the surface tension of pure water at 20 degrees Celsius [3]. The plus or minus sign in Eq. (1) establishes that the Euler equations are bidirectional (waves of each wave number can travel toward both $x = -\infty$ and $x = \infty$ as t increases) as opposed to unidirectional (waves of each wave number travel toward only $x = -\infty$ or $x = \infty$ as t increases). Since the Euler equations are difficult to work with, it is common to introduce the dimensionless parameters

$$\delta = \frac{h_0}{\lambda_0}, \qquad \epsilon = \frac{a_0}{h_0}, \tag{2}$$

in order to derive asymptotic models that are less complicated. Here λ_0 is a typical wavelength and a_0 is a typical wave amplitude. The parameter ϵ is a measure of nonlinearity and the parameter δ is a measure of wavelength or shallowness.

2.1. The KdV equation

The Korteweg–de Vries (KdV) equation can be derived from the Euler equations by assuming that $\delta^2 \sim \epsilon \ll 1$ and truncating at $\mathcal{O}(\epsilon^3)$. In other words, the waves are assumed to have small amplitude and large wavelength. The KdV equation has been well studied mathematically (e.g. [4–7]) and experimentally (e.g. [8–11]). In dimensional form, the KdV equation with surface tension [7] is given by

$$\eta_t + \sqrt{gh_0}\,\eta_x + \frac{3}{2h_0}\sqrt{gh_0}\,\eta\eta_x + \sqrt{gh_0}\Big(\frac{h_0^2}{6} - \frac{\tau}{2g}\Big)\eta_{xxx} = 0, \tag{3}$$

where $\eta = \eta(x,t)$ represents the displacement of the free surface from its undisturbed level. The linear phase velocity for KdV with surface tension is

$$c_K = \sqrt{gh_0}\Big(1 - k^2\big(\frac{h_0^2}{6} - \frac{\tau}{2g}\big)\Big). \tag{4}$$

Fig. 1 contains a plot of the phase velocity for KdV without surface tension (i.e. $\tau = 0$). The plot establishes that KdV only accurately approximates Euler's phase velocity near $kh_0 = 0$ (i.e. in the long-wave limit). This establishes that KdV is a weakly dispersive model. Additionally, for a given k, there is a unique c_K, so KdV is a unidirectional model (even though k such that $|k| < \sqrt{\frac{6g}{gh_0^2 - 3\tau}}$ travel toward $x = \infty$ and k such that $|k| > \sqrt{\frac{6g}{gh_0^2 - 3\tau}}$ travel toward $x = -\infty$).

2.2. The serre equations

The first strongly nonlinear, weakly dispersive set of Boussinesq-type equations was derived by Serre [12,13]. Several years later, Su & Gardner [14] and Green & Naghdi [15] re-derived these equations using different methods. Many have presented rigorous, perturbation theory derivations of the Serre equations, see for example Johnson [16] and Lannes [7].