Wave Motion 79 (2018) 57–72

Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wamot

Travelling wave solutions of the perturbed mKdV equation that represent traffic congestion

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HIGHLIGHTS

- A multi-scale approach is applied to the perturbed mKdV equation.
- Consequently, a family of periodic travelling wave solutions are identified.
- These solutions describe traffic behaviour in an unstable region.
- Their stability is next numerically examined and are shown to be stable.
- Therefore, our travelling waves represent permanent traffic disruptions.

ARTICLE INFO

Article history:
Received 4 January 2017
Received in revised form 13 February 2018
Accepted 22 February 2018
Available online 28 February 2018

Keywords:
mKdV
Modulation theory
Asymptotic solutions
Periodic boundaries
Traffic flow

ABSTRACT

A well-known optimal velocity (OV) model describes vehicle motion along a single lane road, which reduces to a perturbed modified Korteweg–de Vries (mKdV) equation within the unstable regime. Steady travelling wave solutions to this equation are then derived with a multi-scale perturbation technique, where the travelling wave propagation coordinate depends upon slow and fast variables. The leading order solution in the hierarchy is then written in terms of these multi-scaled variables. At the following order, a system of differential equations is highlighted that govern the slowly evolving properties of the leading solution. Next, it is shown that the critical points of this system signify travelling waves without slow variation. As a result, a family of steady waves with constant amplitude and period are identified. When periodic boundary conditions are satisfied, these solutions' parameters, including the wave speed, are associated with the driver's sensitivity, $\hat{a}$, which appears in the OV model. For some given $\hat{a}$, solutions of both an upward and downward form exist, with the downward type corresponding to traffic congestion. Numerical simulations are used to validate the asymptotic analysis and also to examine the long-time behaviour of our solutions.

1. Introduction

To minimise congestion it is necessary to understand traffic behaviour, which has led to many traffic related studies with varied perspectives. Nagatani [1] has given an overview of the different methods that analyse vehicle motion. In general, these techniques can be classified as either a macroscopic or a microscopic approach.

From a macroscopic viewpoint, Lighthill and Whitham [2] and Richards [3] derived a first order nonlinear partial differential equation to characterise traffic density. These workings are now known as LWR theory, however the limitations of this analysis were highlighted by Daganzo [4]. Also, Daganzo [4] discussed certain higher order models that were extensions

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https://doi.org/10.1016/j.wavemoti.2018.02.006
0165-2125/

of this LWR theory. Again, some model failings were identified, including the appearance of 'wrong-way travel'. Zhang [5] later proposed an alternative higher order model without this defect.

Another option is the application of a microscopic model that describes driven-diffusive-systems, such as the KLS model (see Katz et al. [6] and Katz et al. [7]). For instance, Wang et al. [8] used this method to study a vehicle system with multiple lanes. Monte-Carlo simulations and mean field theory were then utilised to establish the traffic dynamics.

The following optimal velocity (OV) model is also an example of a microscopic approach,

$$\frac{d^2x_j}{dt^2} = \hat{a}\left(V(\Delta x_j(t)) - \frac{dx_j}{dt}\right), \quad (1)$$

where $x_j(t)$ is the position of car j at time t, $\Delta x_j = x_{j+1} - x_j$ is car j's headway (the distance between car j and car $j+1$), V is the car's optimal velocity, $j = 0, 1, 2, \ldots, N$ for N cars on the road and $\hat{a}$ is the driver's sensitivity, which is equal to the inverse of the delay time of the driver and vehicle. This equation was derived by Newell [9] and Bando et al. [10] to describe vehicle behaviour on a single lane road. In particular, it ensures that car j accelerates or decelerates in order to achieve a safe distance between itself and the preceding vehicle. The traffic model (1) can be rewritten in terms of the headway such that

$$\frac{d^2\Delta x_j}{dt^2} = \hat{a}\left(V(\Delta x_{j+1}(t)) - V(\Delta x_j(t)) - \frac{d\Delta x_j}{dt}\right). \quad (2)$$

As well, Bando et al. [10] proposed an optimal velocity function, which is of the form

$$V(\Delta x_j(t)) = \tanh(\Delta x_j - h_c) + \tanh(h_c), \quad (3)$$

where h_c is the perceived safe headway distance. This function satisfies the necessary conditions of $V(\Delta x_j(t) = 0) = 0$, V being bounded and $V(h') < V(h_c)$ $(V(h'') > V(h_c))$ for $h' < h_c$ $(h'' > h_c)$. By applying linear stability analysis to (2), a neutral stability line with a critical point is obtained. This line signifies the boundary between two stability regions referred to as metastable and unstable. See Ge et al. [11] for further detail.

Muramatsu and Nagatani [12] reduced (2) to a perturbed Korteweg–de Vries (KdV) equation within the metastable zone using nonlinear theory. This was the KdV equation with higher order correction terms. They then numerically identified traffic solitons propagating over open boundaries, which eventually dissolved. This behaviour is expected within this stability regime since all solutions should tend to the uniform headway. Hattam [13] studied this problem with periodic boundaries, where cnoidal waves were shown to exist that represented traffic congestion. These solutions were derived using modulation theory and then validated with numerical simulations. Again, these density waves disappeared after some time.

In contrast, solutions corresponding to the unstable region were identified by Komatsu and Sasa [14]. Beginning with (2) close to the critical point on the neutral stability line, they derived a perturbed modified KdV (mKdV) equation. The leading order solution to this equation was written in terms of Jacobi elliptic functions that were dependent upon the elliptic modulation term $m \in [0, 1]$. When $m = 1$, this solution became the kink soliton, which exhibits the start/stop motion representative of a traffic jam. Komatsu and Sasa [14] then applied perturbation analysis to seek steady travelling wave solutions of the mKdV traffic model. They established that this solution type only existed when the wave modulus m remained constant and consequently, the wave amplitude and period were fixed. A condition for m as some constant was next found in terms of integral constraints, which then determined the relationship between m and the wave speed. They referred to the travelling wave solutions with $m = 1$ as deformed kink solitons. Otherwise, for constant $m \neq 1$, they were labelled deformed periodic solitons.

Here, a multi-scale perturbation technique is applied to the perturbed mKdV equation to also identify steady travelling wave solutions. This approach is an adaptation of the method outlined by Hattam and Clarke [15] for the steady forced KdV–Burgers equation. Solutions of a similar form to the deformed periodic solitons found by Komatsu and Sasa [14] are highlighted, which satisfy periodic boundaries. Komatsu and Sasa [14] proposed that this solution type was always unstable and only deformed kink solitons were observed numerically. The stability of our periodic waves is investigated here.

Such studies as Zhu and Dai [16] and Zheng et al. [17] have numerically examined OV traffic models within the unstable zone, where periodic boundary conditions were imposed. The long-time behaviour was analysed, which revealed solutions that were indicative of mKdV dynamics as kink-like waves appeared. Moreover, Li et al. [18] performed numerical simulations over large time intervals of an OV model that described a two-lane system with periodic boundaries. As well, this model was transformed into a perturbed mKdV equation near to the critical point. The numerical results corresponding to this region uncovered steady periodic travelling wave solutions with constant amplitude, mean height and period. Hence, these numerical findings suggest stable periodic solutions to the OV traffic system do propagate within this unstable regime. Therefore, additional work is needed to determine the link between the numerical results and the nonlinear theory.

The focus of this paper is the derivation of steady travelling wave solutions to (2) and then the analysis of their long-time dynamics. In Section 2, (2) is reduced to a perturbed mKdV equation and then steady travelling wave solutions are determined using a multi-scale perturbation method in Section 3. The leading order solution is obtained in terms of Jacobi elliptic functions that depend upon slow and fast variables. At the next order, a dynamical system governing the slow variation of the leading order solution is identified. Then, in Section 4, the fixed points of this system are shown to represent a family of steady travelling waves that do not slowly vary. This set of solutions have fixed amplitude, mean height and period. Also, the relations between the solution parameters, the wave speed and the driver's sensitivity are established due to implementing periodic boundary conditions. Lastly, in Section 5, the highlighted periodic asymptotic solutions are compared with numerical results and their long-time behaviour is studied.