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Decay of Benjamin–Ono solitons under the influence of dissipation

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HIGHLIGHTS

- The adiabatic decay of Benjamin–Ono solitons has been calculated for the various kinds of dissipation.
- The exact solution for the Benjamin–Ono equation with the Landau damping has been found.
- Transformation of Benjamin–Ono solitons into the envelope solitons in a rotating fluid has been calculated.

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ABSTRACT

The adiabatic decay of Benjamin–Ono algebraic solitons is studied when the influence of various types of small dissipation and radiative losses due to large scale Coriolis dispersion are taken into consideration. The physically most important dissipations are studied, Rayleigh and Reynolds dissipation, Landau damping, dissipation in a laminar boundary layer and Chezy friction on a rough bottom. The decay laws for the soliton parameters, that is amplitude, velocity and width, are found in analytical form and are compared with the results of direct numerical modelling.

1. Introduction

The propagation of small amplitude long waves in a stratified fluid consisting of a relatively thin layer overlying a very deep passive layer is described by the well-known Benjamin–Ono (BO) equation [1–3]

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \frac{\beta}{\pi} \frac{\partial^2}{\partial x^2} \wp \int_{-\infty}^{+\infty} \frac{u(\xi, t)}{\xi - x} d\xi = 0. \quad (1)$$

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Here, $u(x, t)$ is the perturbation of a pycnocline (a layer with a constant density) and α and β are parameters which depend on the particular stratification (for details see [2,3]). The symbol $\wp$ denotes the principal value of the integral. The BO equation (1) is set in a coordinate frame moving with the speed c of linear long waves. In particular, the BO equation has the algebraic soliton solution [1]

$$u(x,t) = \frac{A}{1+(x-Vt)^2/\Delta^2}. \tag{2}$$

Here, A is the soliton amplitude, $V = \alpha A/4$ is its velocity in the Galilean coordinate frame moving with the speed c with respect to an immovable observer and $\Delta = 4\beta/\alpha A$ is its characteristic width.

The BO equation is similar to the classic Korteweg–de Vries (KdV) equation for weakly nonlinear long waves in a shallow fluid

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0. \tag{3}$$

In particular, both are completely integrable [1] and have families of periodic wave solutions, one limit of which is the solitary wave solution. A specific feature of these solitary wave solutions is that they are robust and restore their parameters (amplitude, shape etc.) after collisions with each other, so that they are termed solitons [4], the only relic of the interaction being a phase shift. They are also stable under arbitrary localised perturbations.

In real physical media there are usually different dissipative mechanisms which affect wave shapes and soliton dynamics. The influence of various types of dissipation on solitons and kinks has been studied in detail based on various model evolution equations (see, for instance, [5]). However, the influence of dissipation on the decay of BO solitons has not been studied as yet. In this paper we fill this gap and study the adiabatic decay of algebraic BO solitons under the influence of weak dissipation of various types; Rayleigh and Reynolds dissipation, Landau damping, dissipation in a laminar boundary layer and Chezy friction when the soliton propagates over rough bottom topography, as well as the dissipation caused by the radiation of small amplitude waves in media containing weak large scale dispersion. Such dispersion typically arises in a rotating fluid [6,7], but can be caused by specific dispersion dependences in other media.

If weak dissipation is taken into account the BO equation (1) is augmented by additional terms whose structure depends on the nature of this dissipation. In general, BO type equations with dissipative terms can be presented in the form

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \frac{\beta}{\pi}\frac{\partial^2}{\partial x^2}\wp\int_{-\infty}^{+\infty}\frac{u(\xi,t)}{\xi - x}d\xi + \delta\mathcal{D}[u] = 0, \tag{4}$$

where $\mathcal{D}[u]$ is an operator which can be expressed in the rather general form [3]

$$\mathcal{D}[u] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}(-ik)^m\tilde{u}(k,t)e^{ikx}dk \quad \text{and} \quad \tilde{u}(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}u(x,t)e^{-ikx}dx. \tag{5}$$

Here $\tilde{u}(k, t)$ is the Fourier transform of the function $u(x, t)$ and the parameter m depends on the specific type of dissipation.

The case $m = 0$ (together with $\delta > 0$) corresponds to linear Rayleigh damping, as the dissipative term $\mathcal{D}[u]$ in Eq. (4) reduces simply to δu. This loss has been invoked in the internal wave context as a model for friction in the bottom boundary layer [5]. Another widely used model in many physical contexts has $m = 2$ and $\delta < 0$. This corresponds to Reynolds dissipation, $\mathcal{D}[u] = -\delta u_{xx}$, and so δ represents the kinematic viscosity of the fluid.

When m is not an even number the term $(-ik)^m$ needs a more careful interpretation. In particular, when $m = 1$ it should be replaced by $|k|$. In this case the dissipative term $\mathcal{D}[u]$ reduces to the Hilbert transform of the derivative u_x and describes the Landau damping of plasma waves, or of internal waves in a stratified fluid with a shear flow [8,9] (see Eq. (26) below for an alternative representation of the BO equation with Landau damping).

For non-integer m the correct interpretation of the term $(-ik)^m$ is

$$(-ik)^m = |k|^m \exp(-i\,\mathrm{sign}\,[k]\,m\,\pi/2)\,, \tag{6}$$

where $\mathrm{sign}\,[k] \equiv k/|k|$. This representation ensures that $\mathcal{D}[u]$ is real valued when u is real valued, as it can be readily verified in this case that $\mathcal{D}[u]^* = \mathcal{D}[u]$, noting that $\tilde{u}^*(k) = \tilde{u}(-k)$ (here the star superscript denotes the complex conjugate). When $m = 1/2$ this dissipative term can be used for the description of wave decay due to a laminar bottom boundary layer [3].

The operator $\mathcal{D}[u]$ can be nonlinear. An important example is $\mathcal{D}[u] = |u|u$ for the dissipation of internal solitary waves over a rough bottom [3]. This model is based on the empirical description of dissipation in a turbulent boundary layer.

Note that for $m > 0$ solutions of the BO equation (4) conserve "mass" $M = \int_{-\infty}^{\infty} u(x,t)\,dx$, whereas for $m = 0$ the total mass is not conserved. This issue has been discussed by Miles [10] in the application of the KdV equation to water waves in a channel which has variable depth and width. In particular, in the case of Rayleigh dissipation, on integrating the perturbed BO equation (4) we obtain the mass balance equation

$$\frac{dM}{dt} = -\delta M\,, \tag{7}$$

which has the solution $M(t) = M_0 e^{-\delta t}$. It is interesting to note that for the BO soliton (2) the total mass $M_s = 4\pi\beta/\alpha$ does not depend on its amplitude. It is then concluded that the mass balance (7) implies that under the influence of even small Rayleigh dissipation the solution $u(x, t)$ cannot be just a BO soliton, but must contain a non-solitonic part (a trailing wave).