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Method of relaxed streamline upwinding for hyperbolic conservation laws

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HIGHLIGHTS

- In this paper relaxation system based new stabilized finite element scheme is proposed for hyperbolic conservation laws.
- Six symmetric discrete velocity models based on orthogonal velocity method are proposed for two-dimensional problems.
- Proposed scheme gives simple diffusion terms required for stabilization of the numerical scheme.
- Proposed scheme can be directly extended from scalar to vector conservation laws.
- Various test cases of Burgers equation, Euler and shallow water equations are solved in one and two dimensions.

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ABSTRACT

In this work a new finite element based Method of Relaxed Streamline Upwinding is proposed to solve hyperbolic conservation laws. Formulation of the proposed scheme is based on relaxation system which replaces hyperbolic conservation laws by semi-linear system with stiff source term also called as relaxation term. The advantage of the semi-linear system is that the nonlinearity in the convection term is pushed towards the source term on right hand side which can be handled with ease. Six symmetric discrete velocity models are introduced in two dimensions which symmetrically spread foot of the characteristics in all four quadrants thereby taking information symmetrically from all directions. Proposed scheme gives exact diffusion vectors which are very simple. Moreover, the formulation is easily extendable from scalar to vector conservation laws. Various test cases are solved for Burgers equation (with convex and non-convex flux functions), Euler equations and shallow water equations in one and two dimensions which demonstrate the robustness and accuracy of the proposed scheme. New test cases are proposed for Burgers equation, Euler and shallow water equations. Exact solution is given for two-dimensional Burgers test case which involves normal discontinuity and series of oblique discontinuities. Error analysis of the proposed scheme shows optimal convergence rate. Moreover, spectral stability analysis gives implicit expression of critical time step.

1. Introduction

Many natural processes are governed by hyperbolic conservation laws like high speed flows governed by compressible Euler equations, shallow water flows like flow in a canal, river *etc* governed by shallow water equations, astrophysical flows or space weather governed by magnetohydrodynamic equations *etc*. These equations describe the transport and propagation of waves (both linear and nonlinear) in space and time. Due to nonlinear nature of convection term such equations admit

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discontinuous solution even when the initial condition is sufficiently smooth. This precludes the possibility of finding closed form solution except in some simple cases. As a last resort, numerical methods are extensively used to solve these equations. Over the past few decades various numerical methods are proposed to solve hyperbolic conservation laws. In the literature of finite volume and finite difference framework, upwind methods are used for solving hyperbolic conservation laws. Upwind methods use biased stencil in the direction determined by the sign of characteristic speeds. These methods are first proposed by Courant et al. in [1]. Various upwind methods are available in the literature like flux splitting methods, flux difference splitting *etc*. Upwind methods are also used in finite element framework where they are part of much larger group called as stabilized finite element methods. There are various stabilized finite element methods available in the literature like Taylor Galerkin method, Streamline-Upwind Petrov–Galerkin method, Discontinuous Galerkin method, Least-Square Galerkin method, Characteristic Galerkin *etc*. A detailed discussion about these methods can be found in [2–6].

Relaxation scheme introduced by Jin & Xin [7] for hyperbolic conservation laws without source term is an attractive alternative to the upwind schemes. In recent years, the simplicity of this scheme attracted many researchers around the world. Relaxation scheme is based on the relaxation system which replaces nonlinear convection term present in hyperbolic partial differential equation(s) by semi-linear system with stiff relaxation term on right hand side. This system is equivalent to original hyperbolic conservation law in the limit of vanishing relaxation parameter. Simple procedure present in Relaxation schemes to handle nonlinear convection term avoids more complex Riemann solvers. First and second order relaxation schemes are first introduced in [7] while higher order relaxation schemes are discussed in [8–10]. Relaxation schemes for hyperbolic conservation laws with source term are presented in [11–13]. Natalini [14] interpreted Jin & Xin's relaxation system as a discrete velocity Boltzmann equation with BGK model for the collision term. In [15] Aregba-Driollet and Natalini introduced numerical schemes based on discrete velocity Boltzmann equation which are called as discrete velocity kinetic schemes. Relaxation schemes are also employed in the lattice Boltzmann framework [16,17]. An alternative relaxation system for one-dimensional conservation law proposed by Murthy [18] retain the semi-linear structure of original relaxation system but at the same time satisfies the integral constraint which is more consistent than the original relaxation system. For more details on relaxation schemes refer [7,14,15,19–25] and the references there in.

Relaxation scheme is also used in finite element framework for one dimensional scalar and vector (elastodynamics) problems [26]. In this paper the relaxation system based Streamline-Upwind scheme, named as Method of Relaxed Streamline Upwinding (MRSU) is developed in finite element framework for hyperbolic conservation laws in one and two dimensions. The proposed scheme belongs to the class of stabilized finite element methods. Some of the salient features of the proposed formulation are

1. Weak formulation of governing hyperbolic conservation laws in conservation form is obtained from relaxation system assuming instantaneous relaxation to equilibrium.
2. Only test space of convection part of the governing equation is enriched to obtain the required stabilization term.
3. Group discretization also called as group formulation which is shown to be more accurate is used for flux function [27,28].
4. Proposed scheme can be easily extended from scalar to vector conservation laws.
5. Exact stabilization terms are obtained for both scalar as well as vector conservation laws. Importantly, computationally expensive Jacobian matrices are not involved in the stabilization terms.
6. Six symmetric discrete velocity models are introduced in two dimensions which include four points along diagonal ($\mathscr{D}4$), nine points including rest particle ($\mathscr{A}\mathscr{D}9$), eight points without rest particle ($\mathscr{A}\mathscr{D}8$), four points along axis ($\mathscr{A}4$), four points along diagonal with one rest particle ($\mathscr{D}5$) and four points along axis with one rest particle ($\mathscr{A}5$) discrete velocity models.
7. To show the efficacy of the proposed scheme various test cases of Burgers equation, Euler equations and shallow water equations are solved in one and two dimensions. Moreover, some new test cases of Burgers equation, Euler and shallow water equations are also introduced. In case of two-dimensional Burgers equation a set of test cases involving normal and oblique discontinuity is proposed along with their exact solution.
8. Error analysis of the proposed scheme shows optimal rate of convergence. Furthermore, spectral stability analysis gives implicit expression of critical time step.

This paper is arranged as follows. After introduction in Section 1, Section 2 describes governing hyperbolic conservation laws like Burgers equation, Euler and shallow water equations. Section 3 gives relaxation system for hyperbolic conservation laws. In Section 4, relaxed formulation of hyperbolic conservation laws is explained which will be used to develop MRSU scheme. Section 5 explains two and three discrete velocity models for one-dimensional problems whereas Section 6 describes various symmetric discrete velocity models for two-dimensional problems. In Section 7, Chapman–Enskog type expansion of relaxation system is performed which gives stability condition for such system. Section 8 gives weak MRSU formulation in detail. Section 9 describes temporal discretization of semi-discrete MRSU scheme followed by Section 10 where simple gradient based shock capturing parameter is defined. In Section 11, spectral stability analysis of the proposed scheme is carried out which gives expression of critical time step. In Section 12, many numerical experiments are carried out for Burgers equation, Euler and shallow water equations which support author's claim of robustness and accuracy in the proposed numerical scheme. Finally, this paper is concluded in Section 13.