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Fluid–fluid and –solid interaction problems: Variational principles revisited

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ABSTRACT

We systematically review some unified variational principles for a strong interaction problem in both a stratified fluid region and a fluid–solid region. The problem is described by a general Lagrangian formulation for an anisotropic elastic solid region, either surrounded by an incompressible non-Newtonian fluid region or surrounding the fluid region. In the first part, we express the fundamental equations of the regular fluid and solid regions in differential form. Then, we deduce the variational principles respectively from the principle of virtual power and the principle of virtual work for the fluid and solid regions. The physics principles are modified through an involutory transformation together with a dislocation potential. In the second part, we similarly establish some multi-field variational principles for a stratified fluid of two or more distinct fluid layers of different thicknesses and mass densities. In the third part, we derive the variational principles for the interior and exterior interaction problems in a fluid region with a surface piercing solid, within either a rigid or an elastic structure. The variational principles, which operate on all the field variables lead to the fundamental equations of the regions, including the interface conditions, as their Euler–Lagrange equations. Some special cases of the variational principles are given.

1. Introduction

In mechanics, the physical response of a continuum may be mathematically modelled by the fundamental equations, which consist of the divergence and gradient equations, the constitutive relations, and the boundary and initial conditions to meet the internal consistency (i.e., uniqueness and existence) of their solutions. The divergence equations were originally established in integral (global) form by the simultaneous application of the axioms of continuum mechanics, and also, in differential form under certain regularity and local differentiability conditions of the field variables. The constitutive relations were always given in differential form under certain rules and invariant requirements of continuum mechanics, except for the non-local case, as generally were the boundary and initial conditions. Alternatively, some or all the fundamental equations can be stated in variational form, as the Euler–Lagrange equations of a variational principle, which is most often desirable, as a standard means in solving directly the initial and mixed boundary value problems. Besides, the variational principles were used in examining the internal consistency of solutions, in finding the bounds formulae and in establishing the lower order equations of continuum mechanics. Historically, a panoramic development of energetic and related variational principles in mechanics, which may be traced back to Heraclitos of Ephesos and others of Hellenic Science was reported (e.g., [1,2]). In fluid mechanics, Lord Kelvin [3] reported the first classical variational principle, which applies to ideal fluids, and following his minimum energy principle, many researchers (e.g., [4–12]) formulated various variational principles of fluids. In solid mechanics, Prange [13,14] and Hellinger [15] who were inspired by Hilbert's lectures on

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Nomenclature

Ξ	Euclidean 3-D space
θ^i	fixed, right-handed general system of curvilinear coordinates in Ξ
t, $T = [t_0, t_1)$	time, time interval
$\dot{\chi}, \chi_{i,j}, \chi_{i;j}$	time differentiation, and partial and covariant differentiation with respect to θ^i
g^{ij}	metric tensor
e_{ijk}	alternating tensor
$\Omega(t)$, $\Omega \times T$	region at time t, Cartesian product of the region and the time interval
$\Omega + \partial\Omega, \overline{\Omega}$	regular, finite and bounded region, its boundary surface and closure (=$\Omega + \partial\Omega$)
S	free surface of a fluid region
n_i, v_i, N_i, η_i, ς_i	unit outward vectors normal to the boundary surfaces
u_i, a_i	displacement and acceleration vectors
v_i	velocity vector
ρ, f_i	mass density, body force vector
f, s, m	terms belonging to fluid, solid and multi-layer (stratified) fluid
p	hydrodynamic pressure
τ^i, τ^{ij}	viscous traction vector and viscous stress tensors
σ^i, σ^{ij}	traction vector and stress tensor in fluid region
t^i, t^{ij}	traction vector and stress tensor in solid region
d_{ij}, e_{ij}	strain rate and strain tensor
f, ff, s	stand for fluid and stratified (multi-layer) fluid regions, and solid region
I_c, I, J	incompressibility, initial and interface condition terms
D, G, C, B	divergence, gradient, constitutive and boundary terms
c^{ijkl}, λ, μ	material constants
Λ	admissible state
Δ	dislocation potential
λ^i, λ^{ij}	Lagrange multipliers
$C_{\alpha\beta}$	functions with derivatives up to and including α and β with respect to θ^i and t
χ^*, $\langle\chi\rangle$	prescribed and mean value of $\chi = \frac{1}{2}(\chi_{(1)} + \chi_{(2)})$
$[\chi]$	jump of $\chi (= \frac{1}{2}(\chi^{(2)} - \chi^{(1)})$

mechanics of continua and calculus of variations constructed the first classical variational principles of elastodynamics, and then, a large number of authors (e.g., [16–19]) contributed to the subject as well. A rich number of variational principles were derived in almost all the fields of mechanics (e.g., hygro-thermo-piezoelectric fields [20,21]), and an elaborate account of variational principles in fluid and solid mechanics, with their development and extensive applications can be found in the treatises (e.g., [22–33]).

The variational principles were either formulated by a number of mathematical methods (e.g., the method of convolution due to Gurtin [34] or deduced from a general principle of physics (e.g., Hamilton's principle, the principle of virtual work, and the principle of virtual power) and/or by extending it through a method of relaxation (e.g., [35]). With the application of a mathematical method, an integral type of variational principles with an explicit functional can be derived but only for a linear and self-adjoint system of differential equations, and its existence can be tested by use of Fréchet derivatives [36]. A physics principle denies, by definition, a functional due to its postulated statements in terms of infinitesimals (e.g., virtual displacements/velocities, and virtual work). Accordingly, the physics principle always leads to a differential type of variational principles without a functional, which can be almost always established for any system of differential equations. Among the physics principles, Hamilton's principle admits an explicit functional in the presence of conservative forces only. Moreover, the variational principles deduced from a physics principle generate only the divergence equations and the associated Neumann's type of boundary conditions, as their Euler–Lagrange equations. Thus, the rest of the fundamental equations remain as the constraint (subsidiary) conditions, which are most often undesirable in computation. However, the constraint conditions can be relaxed by, for instance, an involutory (Friedrichs's, Legendre) transformation (e.g., [37,38]), which was widely used due to its versatility and rather easy application to holonomic as well as non-holonomic constraint conditions (e.g., [39]). In this paper, we deduce some variational principles for the linear problems of a fluid–fluid (stratified) and a fluid–solid (structure) strong interaction from the general principles of physics by modifying them through the involutory transformation.

The fluid–solid (fluid) interaction problems gained increasingly a lot of research interest during the last few decades, due to their immense technological applications not only in various branches of engineering but also in medical sciences, in the context of fluid and solid mechanics (e.g., [40–51]). In an interaction problem, either a solid region is immersed partly or fully in a fluid region of finite (or infinite) extent (i.e., the exterior problem) or surrounding a fluid region (i.e., the interior problem), and the regions influence strongly one another in many instances and their interaction becomes significant in their physical response. The strong interaction is of primary concern in the case of a heavy fluid (e.g., water) or a rather light solid