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Three-dimensional overturned traveling water waves

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HIGHLIGHTS

- A system of equations for traveling waves on parameterized surfaces is developed.
- Three dimensional traveling waves are computed via numerical continuation.
- An example of a three-dimensional traveling wave with overturned crests is presented.
- The structure of dimension-breaking bifurcations is investigated.

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ABSTRACT

Traveling gravity-capillary water waves on the interface of a three-dimensional fluid of infinite depth are computed. The vortex sheet formulation with the small scale approximation is used as the mathematical model for the fluid motion. The fluid interface is parameterized isothermally. The traveling wave ansatz for parameterized surfaces is described. Waves are computed using Fourier collocation and quasi-Newton iteration; large amplitude overturned traveling waves are computed via a dimension-breaking based numerical continuation method.

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1. Introduction

We study periodic waves of the interface between two constant-density fluids undergoing irrotational motions. The fluid regions are infinitely deep in the vertical direction and periodic in the horizontal direction. We seek traveling wave solutions, in which the free surface is of permanent form and steadily translating. This study is fundamentally concerned with waves on a two-dimensional interface, between three-dimensional fluids, which may have overhanging crests (or troughs).

It is the understanding of the authors that no study has been conducted for fully three-dimensional water waves which are both overturned and traveling. A number of studies have considered overturning in the time dependent problem, for example [1–7] with a review in [8]. There are also numerous computations of permanent three-dimensional waves (both traveling and standing waves) in which the interface is parameterized by the horizontal coordinates, for example [9–14]. There have been studies of axisymmetric three-dimensional overturned traveling waves in fluid jets, where such symmetry is natural [15–17].

The reasons for the absence of previous work on three-dimensional overturned traveling waves are two-fold. First, one must have a three-dimensional formulation of the problem which allows for traveling waves which are overturning. Conformal mappings are by far the most popular technique for the two-dimensional problem, but do not generalize to three-dimensions. In a recent work, the first author and collaborators have developed a formulation which extends to three-dimensions and allows for the computation of traveling waves on interfaces with arbitrary parameterizations [18]. It is in

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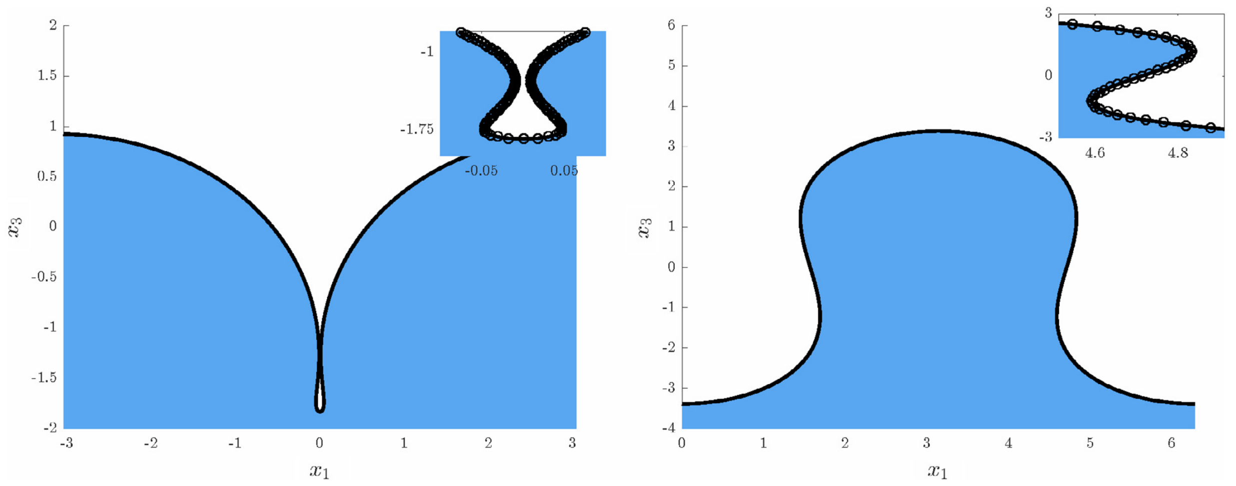

Fig. 1.1. The extreme water wave on branches of traveling waves at two different Bond numbers in a two-dimensional fluid are depicted. Left: The large amplitude limit of traveling waves with $\sigma = 1/8$ is a self-intersecting profile. Right: The steepest wave for $\sigma = -1/10$. The left panel was computed with $M_a = 512$ points, the right panel with $M_a = 128$. The increased resolution on the left is to resolve the extreme curvature within the bubble. In both panels the inset figures show close-ups of the overturned portion of the wave, with grid-points marked with circles.

this formulation that this paper proceeds to three-dimensions. The need for such parametric formulations of the water wave problem is not unknown. Alternative to the track taken here, Bridges and Dias proposed a Hamiltonian formulation which allows for arbitrary interface parameterizations [19].

The second reason for the lack of computations of overhanging three-dimensional traveling waves is the extreme expense of the computation itself, as will be discussed explicitly here, and is reviewed in [8]. In this work, the extreme cost will be partially ameliorated via the use of an approximate model, called the small-scale approximation, proposed in [20] and later used in [21]. The approximation allows the most costly part of the computation, the evaluation of the Birkhoff–Rott integral, to be computed via fast Fourier transforms. The small scale approximation, although exact in the small-amplitude limit, is not based on a small-amplitude assumption, and will be used here to compute large amplitude three-dimensional traveling waves, including those with overturned crests and troughs.

For two-dimensional fluids, a significant amount of work has been done in the study of both dynamic and steady overturned waves. We will not try to review them all here. Most relevant to this work are the exact traveling solutions of Crapper [22] and the numerically computed waves of Meiron and Saffman [23], as these two waves are qualitatively similar to the cross-sections of the three-dimensional waves computed here. This paper also is an outgrowth of a number of recent two-dimensional studies by one of the authors. The traveling wave ansatz developed in [18] has since been used extensively to compute two-dimensional overturning traveling waves [24,25].

The remainder of the paper is organized as follows. In Section 2 we present the vortex sheet formulation of the potential flow equations, the small scale approximation to the Birkhoff–Rott equations, and the traveling wave ansatz. These three ingredients combine to give the system of equations which are solved for three-dimensional traveling waves. In Section 3 we present the numerical procedure used to compute traveling waves as well as the numerical results. This Section 3 includes an example of an overturned three-dimensional traveling wave and discussion of the dimension-breaking continuation procedure used to compute three-dimensional waves. In Section 4 we summarize our results and present future research avenues.

2. Formulation

In this work we compute three-dimensional traveling waves in a model for the interface between two-fluids. In particular, we are interested in the case where the fluid interface is overturned, that is, where the vertical displacement is not a function of horizontal Cartesian coordinates. To compute such three-dimensional overturning waves, we will represent the interface as a parameterized surface $\vec{X}(\alpha, \beta, t) = (x_1(\alpha, \beta, t), x_2(\alpha, \beta, t), x_3(\alpha, \beta, t))$. Following Ambrose, Siegel and Tlupova, [1], we will enforce that this parameterization is isothermal, i.e. that

$$\vec{X}_\alpha \cdot \vec{X}_\beta = 0, \quad \text{and} \quad G \equiv \|\vec{X}_\alpha\|^2 = \lambda \|\vec{X}_\beta\|^2 \equiv \lambda E \tag{2.1}$$

with

$$\lambda = \frac{\iint G \, d\alpha d\beta}{\iint E \, d\alpha d\beta}.$$

We will think of λ as a constant specified at the beginning, describing the aspect ratio of the parameterization (or how much longer the wave is in α than in β). We choose to set $\lambda = 1$, so that $G = E$. We set the ranges for α and β to be equal to the