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An analytical–numerical method for fast evaluation of probability densities for transient solutions of nonlinear Itô's stochastic differential equations

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ABSTRACT

Probability densities for solutions of nonlinear Itô's stochastic differential equations are described by the corresponding Kolmogorov-forward/Fokker–Planck equations. The densities provide the most complete information on the related probability distributions. This is an advantage crucial in many applications such as modelling floating structures under the stochastic-load due to wind or sea waves. Practical methods for numerical solution of the probability density equations are combined, analytical–numerical techniques. The present work develops a new analytical–numerical approach, the successive-transition (ST) method, which is a version of the path-integration (PI) method. The ST technique is based on an analytical approximation for the transition probability density. It enables the PI approach to explicitly allow for the damping matrix in the approximation. This is achieved by extending another method, introduced earlier for bistable nonlinear reaction–diffusion equations, to the probability density equations. The ST method also includes a control for the size of the time-step. The overall accuracy of the ST method can be tested on various nonlinear examples. One such example is proposed. It is one-dimensional nonlinear Itô's equation describing the velocity of a ship maneuvering along a straight line under the action of the stochastic drag due to wind or sea waves. Another problem in marine engineering, the rolling of a ship up to its possible capsizing is also discussed in connection with the complicated damping matrix picture. The work suggests a few directions for future research.

1. Introduction

Many phenomena analyzed in the natural sciences and engineering are described by solutions of nonlinear Itô's stochastic differential equations (ISDEs) (e.g., [1–3]). In particular, may be specific applications refer to nonlinear behavior of ships and ocean structures under the stochastic-load effect due to sea waves or wind (e.g., [4–8]). A fairly general stochastic treatment describing sea waves with a random field can be found in [3, Chapter 8]. Minoura and Naito [9,10] present the results on the ISDE-based modelling of see waves which are less general but better suited to practical analysis.

The nonlinear behavior of floating structures in the stochastic sea waves or wind is associated with stochastic systems which include components with both lumped and distributed characteristics. This area is one of many examples of engineering

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fields that can properly be dealt with only by means of a fairly capable and flexible stochastic-modelling paradigm. An advanced treatment of this kind was developed by Bellomo and co-authors for a number of years. The resulting, stochastic-continuum-mechanics paradigm comprises:

- approaches to partial integro-differential equations which are fundamental in continuum-mechanics, the Boltzmann-type equations (e.g., [11]);
- a fairly universal and efficient computational technique for Itô's stochastic partial (integro-)differential equations which, in certain respects, present a generalization of the aforementioned models, the so-called stochastic-adaptive-interpolation (SAI) method [12] (see also [13, Chapter 5] on the version with bases in functional Banach spaces);
- a consistent extension of the above treatments to especially complicated problems in living matter such as problems in biology, medicine, and social sciences (e.g., [14,15]).

In the present context, the key advantages of the SAI technique are the following:

(i) It reduces a single Itô's stochastic partial differential equation to a system of ISDEs where equations correspond to the space-discretization points. This not only enables one to apply the well-known methods of ISDE theory but also admits to seamlessly combine the terms for the lumped- and distributed-characteristic components in a unified computational model.
(ii) The SAI method is a stochastic generalization of the Bellman differential-quadrature technique, and, consequently, inherits many attractive features of the latter. One of them is the fact that the number of the space-discretization points can usually be much less than that in other numerical methods at the same values of the achieved accuracy (e.g., see [13, Section 5.4] for the discussion and further references]). Owing to that, the dimensions of the state space for the ISDE systems resulting from the SAI-method-discretization of Itô's stochastic partial (integro-)differential equations is normally rather low.

These properties indicate that the stochastic-continuum-mechanics treatment developed by Bellomo and co-authors presents an advantageous choice as the unified analysis environment in engineering sciences. Properties (i) and (ii) also draw attention to the fact that this treatment allows to reduce complex, continuum-mechanics models to ISDE systems thereby emphasizing the importance of further development of the techniques for the latter models.

Solutions of ISDEs are usually (e.g., [2, Section 9.3]) diffusion stochastic processes (which are, by definition, also Markov processes). In many cases, the probability distributions of these solutions can be studied in terms of the transition and initial probability densities. The initial density is normally a part of the input data for the problem, whereas the transition density is determined as the fundamental solution of the corresponding Kolmogorov-forward/Fokker–Planck (KF/FP) equation. The latter is a linear equation with partial derivatives where the independent scalar variables are the time and the states of the object modelled with an ISDE. If, say, d is the number of these states, then the KF/FP equation is the evolution equation in the d-dimensional space.

The probability distributions presented with the densities provide the complete probabilistic descriptions of the underlying stochastic processes. For this reason, the availability of the transition density is highly desirable in many engineering problems. However, this density can be determined analytically only if the KF/FP equation granting it corresponds to the ISDEs of very particular forms, for instance, when the ISDE is linear. That is why one usually obtains the transition density by means of numerical solving the KF/FP equation. Regrettably, this solving takes large computing resources, even if d is one or two, and seems to be very difficult if d exceeds four. For instance, as noted in [13, p. 203], numerical solutions of nonstationary parabolic partial differential equations were achieved in [16] at $d = 5$ and $d = 6$ only when the forms of the equations were quite particular. Thus, in general, the KF/FP equation approach is applicable if d is fairly low. However, the capabilities of modern computer hardware and computing techniques can make this approach suitable to the ISDE systems which result from the SAI-method (see Property (ii) in Section 1) and correspond to not very high d.

Works [17–20] develop the analytical–numerical path-integration (PI) method developed specifically for the KF/FP equations. It employs an analytical approximation for the transition probability density at each time step of the numerical procedure. This allows to obtain sufficiently accurate numerical solutions of the equations for low computing time. However, the above approximation does not explicitly take into account the damping intensity determined by the drift coefficients of the corresponding ISDEs and represented with the damping matrix.

A technique somewhat resembling the PI method was introduced in [21, Appendix C] and applied to simulation of oncogeny [22]. This technique, the time-slice (TS) method, was developed for bistable nonlinear reaction–diffusion equations. It performs equally efficient in both the stability and instability regions of the solutions. This advantage is due to the special feature of the TS method that it takes into account the damping-related effects which can, in terms of the KF/FP equations, be interpreted by means of the aforementioned drift coefficients.

The purpose of the present work is development of an analytical–numerical method for the KF/FP equations which would be a sort of bridge between the PI and TS techniques in the following sense. It would endow the transition-probability approximations employed in the former one with the damping-effect capabilities inherent in the latter one. This will in principle improve the approximation accuracy and provide a more consistent account of the damping features. Notably, the