Wave Motion 68 (2017) 283–290

Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wavemoti

Mechanical wave momentum from the first principles

Leonid I. Slepyan

School of Mechanical Engineering, Tel Aviv University, P.O. Box 39040, Ramat Aviv 69978 Tel Aviv, Israel

HIGHLIGHTS

- Momentum carrying by a wave as a product of the wave mass and speed.
- Binary waves as and self-equilibrated momentum.
- Momentum of longitudinal and flexural waves.
- Physical sense of the so-called "wave momentum".

ARTICLE INFO

Article history:
Received 18 May 2016
Received in revised form 31 October 2016
Accepted 3 November 2016
Available online 8 November 2016

Keywords:
A. Dynamics
B. Stress waves
Wave mass
C. Asymptotic analysis

ABSTRACT

Axial momentum carried by waves in a uniform waveguide is considered based on the conservation laws and a kind of the causality principle. Specifically, we examine (without resorting to constitutive data) steady-state waves of an arbitrary shape, periodic waves which speed differs from the speed of its form and binary waves carrying self-equilibrated momentum. The approach allows us to represent momentum as a product of the *wave mass* and the wave speed. The propagating wave mass, positive or negative, is the excess of that in the wave over its initial value. This general representation is valid for mechanical waves of arbitrary nature and intensity. The finite-amplitude longitudinal and periodic transverse waves are examined in more detail. It is shown in particular, that the transverse excitation of a string or an elastic beam results in the binary wave. The closed-form expressions for the drift in these waves functionally reduce to the Stokes' drift in surface water waves (a half the latter by the value). Besides, based on the general representation an energy–momentum relation is discussed and the physical meaning of the so-called "wave momentum" is clarified.

1. Introduction

Along with energy, momentum plays a defining role in wave actions, and different related questions were debated since Lord Rayleigh's works on the theory [1,2]. Brillouin [3], McIntyre [4], Ostrovsky and Potapov [5], Peskin [6], Falkovich [7] and Maugin and Rousseau [8] are among others who considered various aspects of this topic (a comprehensive list of references can be found in the latter book by Maugin and Rousseau).

Note, however, that momentum, in contrast to the energy, generally is not a nonnegative value and it is very sensitive to the formulation. In particular, it may be lost with linearization. By definition, momentum is the product of density and the particle velocity, $p = \varrho \boldsymbol{v}$, and (possibly small) variations of both of them should be taken into account. The question is whether it is possible to represent the quantity in a classical manner as the product of a mass and the wave velocity, where only one of the multipliers, the mass can be variable. Below, we introduce such a representation.

E-mail address: leonid@eng.tau.ac.il.

http://dx.doi.org/10.1016/j.wavemoti.2016.11.005
0165-2125/

Most known results correspond to specific problems and low-intensity waves. The above-discussed representation results in a straightforward way for determining the momentum of high-intensity waves. In dispersive sinusoidal waves, the energy propagates with the group velocity. Does the latter play a similar role for momentum? We show that affirmative reply is correct.

Under a linearized formulation where momentum is lost, could it be possible to extract it based on the simplified description of the wave? We show that such a possibility does exist, and the lost momentum can be obtained considering the mass distribution or even only the waveform (as, for example, in the case of the flexural wave considered below).

In the wave theory, along with the classical definition of momentum, the so-called "wave momentum" $-\varrho_0 u' v$ is considered (ϱ_0 is the initial density, u' and v are the derivative of the displacement and the particle speed). It is "... a much debated notion" [8, p. 2]. "Of course, considering a "definition" of wave momentum... would be quite reasonable from a strict mathematical viewpoint, but the physical meaning would be doubtful". (ibid, p.14). Below, based on the general representation the physical meaning of the "wave momentum" is given.

In the present work, we show that the essence of the issue and useful relations immediately follow from the mass and momentum conservation with the assumption that ahead of the wave the waveguide is in a uniform static state until the wave arrives (this is a kind of the *causality principle*). Abstracting from constitutive relations and other particularities a greater clarity can be retrieved, and the results obtained directly from these first principles are valid for mechanical waves of any nature, form and intensity.

We consider a homogeneous straight-line waveguide of an arbitrary cross-section and three types of mechanical waves: a steady-state wave of an arbitrary shape, a periodic wave which speed differs from the speed of its form (for a linear sinusoidal wave these are the group and phase velocities, respectively) and a binary wave carrying a self-equilibrated momentum.

The momentum follows as a product of the *wave mass* and the wave speed. Based on the general representation the wave-induced drift and the connection between the momentum and energy are discussed. Periodic longitudinal and flexural waves are considered as nontrivial examples.

Along with the *reference frame* associated with this initial state, we use a frame moving with the wave. Eulerian (spatial) variables are used, unless otherwise is specified. We denote the longitudinal coordinate by x. The mass and momentum densities per unit x-length and the x-component of the particle velocity are denoted by $m = m(x, t)$, $p = p(x, t)$ and $v = v(x, t)$, respectively. In the initial state of the waveguide, ahead of the wave, $m = m_0$, $v = 0$ (or asymptotically equal to these quantities as $x \to \infty$). As usual, the prime and dot mean the partial derivatives on the coordinate x and time t.

2. Steady-state wave

We are interested in axial momentum of a wave of a general form and intensity propagating along the x-axis with the speed c (note that the wave can possess both the axial and angular momenta, see, e.g., [9]). In the frame moving with the wave, the mass distribution is assumed to be fixed over the wave domain as well as ahead of the wave where the uniform waveguide is in the initial static state (condition $\mathcal{A}$).

2.1. The wave mass and momentum

As the first state consider the wave in the moving frame, where the waveguide medium flows through the wave domain, and momentum in front of the wave is equal to $-m_0 c$. For such a steady movement, it follows from the mass conservation and the above condition $\mathcal{A}$ that

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\int_S \varrho v \, \mathrm{d}S = -\frac{\partial m}{\partial t} = 0 \quad \Longrightarrow \quad p = p_- = -m_0 c, \tag{1}$$

where ϱ and v are density and the x-component of the particle velocity, and S is the cross-section area. Now, adding the corresponding right-directed rigid momentum $p_+ = mc$ (to return to the reference frame) we obtain the momentum density

$$p = m_w c, \quad m_w = m - m_0. \tag{2}$$

The excess of the mass in the wave m_w can be called the wave mass (per unit length). It does not consist of the same particles but propagates as the wave. Note that both these parameters, the wave mass and momentum, can be positive, zero or negative. In the latter case, the momentum directs opposite to the wave.

A step wave is the simplest example. The mass conservation reads as

$$\varrho(c - v) = \varrho_0 c \quad \Longrightarrow \quad p = \varrho v = (\varrho - \varrho_0)c. \tag{3}$$

Recall that v is the particle velocity and ϱ and ϱ_0 are the actual and initial mass densities.

The advantage of the representation (2) as compared with the initial definition, $p = mv$, is that it contains only one variable m. In certain cases, this allows immediately to indicate the presence of momentum and to determine it based on the linearized expression of the wave, as in the case of a flexural wave considered below.

An arbitrary transverse wave in an inextensible string under a constant tensile force T is a simple example. The above relations are valid for this case, and the wave mass corresponding to an x-segment is $\varrho_0(\ell_s - \ell)$, where ℓ_s is the arc length and ℓ is the segment length.